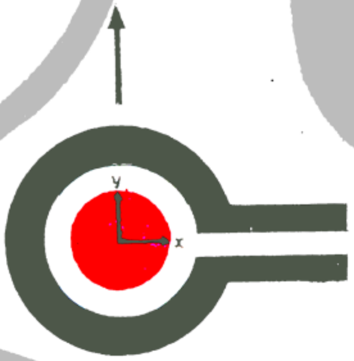
A. I. Kalandiya

Mathematical methods of two-dimensional elasticity



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The book is devoted to systematic presentation of new results obtained on the basis of further development of the widely used methods of N. I. Muskhelishvili. It expounds the formulations and solutions of a number of two-dimensional problems important both from a theoretical and a practical viewpoint, namely, problems of the plane deformation and bending of plates involving the effect of holes, inclusions and stiffeners. Much attention is given to the development and application of efficient approximate methods with the use of electronic computers.

The book is intended for scientific workers and research engineers whose interest is in the mechanics of deformable solids and structural mechanics. It will also be found useful by postgraduates and advanced undergraduate students majoring in these fields. The reader is assumed to be familiar with the fundamentals of the mathematical theory of elasticity, the elements of the theory of functions of a complex variable and (for reading the first chapter) the basic principles of the theory of one-dimensional singular integral equations.

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ИЗДАТЕЛЬСТВО
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А. И. Каландия

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A. I. Kalandiya

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Dedicated to Nikolai Ivanovich Muskhelishvili

Preface

The book is concerned with the methods of solving two-dimensional problems of elasticity theory based on the application of the techniques of complex function theory. The most important and fundamental of these methods are associated with the name of N. I. Muskhelishvili and are presented in his classical monograph "Some Basic Problems of the Mathematical Theory of Elasticity" which saw the fifth Russian edition recently. Owing to the elegance of the analytical means and also to an extreme convenience in applications these methods have become very popular and are continually used by a variety of specialists engaged in widely different fields of mathematical physics from the theory of elasticity and hydrodynamics to quantum field theory.

The idea of applying the complex variable to the solution of the biharmonic equation conceived at the beginning of our century was subsequently extensively used and fundamentally developed in various directions. It is on this basis that a new promising approach has been found to the elucidation of important structural properties of the solution of a wide class of elliptic equations and to the investigation of the associated boundary value problems. On the other hand, many of the followers of N. I. Muskhelishvili have conducted, particularly from the early forties, intensive work on the application and development of the existing methods for the solution of basic problems of the theory of elasticity with the purpose of adapting them for an effective treatment of a class of new problems, most important for applications. A considerable progress in computer mathematics observed during the last decades due to the appearance of high-speed electronic computers has made it possible to represent the solution of many of these problems in a form well suited for purposes of engineering practice and thereby to fulfil finally the wish expressed by A. N. Krylov in his famous preface to the first edition of N. I. Muskhelishvili's book. Of particular note in this direction are the efforts of the Soviet and American scientists.

The author considers the book proposed to the reader's attention as his contribution to the development of this general direction. The book, a more complete idea of whose content may be got from the detailed table of contents, is based, with a few exceptions, on the results of the author's work carried out at the Tbilisi A. M. Razmadze Mathematical Institute and the Computing Centre of the Georgian SSR Academy of Sciences. Apart from Chapter One devoted to a theoretical investigation of mixed problems, the book proposes effective solutions of various stress problems for two-dimensional homogeneous and heterogeneous elastic bodies relating to plane strain, generalized plane stress and also to the theory of bending of thin plates under normal forces.

Chapter Two illustrates, in three examples of the non-standard type, the application of Muskhelishvili's methods to the solution, in a rigorous mathematical sense, of problems which do not admit the derivation of a solution in closed form, with the construction at the same time of an approximate solution with any specified accuracy.

Chapter Three proposes a method for the solution of problems of contact between two bodies with cylindrical surfaces when the line of contact is a circular arc and no assumption is made as to the smallness of the segment of contact.

Chapter Four, which considerably exceeds by volume all the others, proposes special methods for the solution of various kinds of stress problems, adapted for one or another class of regions.

Chapter Five, the last one, stands somewhat by itself. Here an attempt is made to apply the complex variable to the theory of unsymmetrical elasticity and the solution is given for the corresponding boundary value problems in some simple cases.

The presentation is almost universally accompanied by specific examples whose solution is given in explicit analytic form and in a number of cases is brought up to numerical results. The results of Sections 16, 29, 31, 34, 38, 39, 40, 41, 47, 49 are published for the first time.

The questions of adequacy of the derived solutions from the viewpoint of the applicability of the basic relations of linear theory are not discussed in the book for all their importance. The author saw his principal purpose in demonstrating the effectiveness of the methods used, and their detailed presentation received primary attention.

The author wishes to express appreciation to his friends and colleagues, both mathematicians and mechanicians, for helpful discussion of various problems dealt with in the book and invariable attention and interest in his work. Most particular thanks are due to N. N. Dzhgarkava, engineer-programmer, for her great help in performing the computations.

A. I. Kalandiya

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CHAPTER ONE

Method of Integral Equations (Plane Mixed Problems)

This chapter is concerned with the study of two-dimensional problems of the theory of elasticity with mixed boundary conditions by the method of integral equations.

The method of integral equations, one of the basic methods of classical mathematical physics, was widely popular at one time. Immediately after the appearance of the first fundamental works this method attracted the attention of a variety of scientists and became one of the basic methods for the theoretical investigation of boundary value problems. Especial enthusiasm for this method in the study of problems of the mathematical theory of elasticity was observed in the thirties and early forties when Fredholm's integral equations were derived for nearly all basic cases.

The method of potential theory is, of course, a great convenience for general investigations of boundary value problems. However, it is not very suitable for practical applications. The properties of the kernels of potentials which enable one to reduce the boundary value problem to the equation of the second kind render these potentials almost useless for a quantitative analysis of the particular phenomenon under study near the boundary. For practical purposes the method of integral equations with its standard computational scheme based on the use of a discrete analogue of the integral equation is ineffective even with the present level of computational techniques. Except in very few cases, the potential method only provides theorems of uniqueness of solutions, and this is apparently not the chief thing in classical problems of the applied nature, in particular in problems of the theory of elasticity. In such cases, therefore, one has to search for special methods of (effective) solution adapted to specific classes of problems.

The proposed investigation of mixed problems is, in our opinion, of some theoretical interest.

The point is that mixed problems lead in a natural way to integral equations with kernels of the Cauchy type which have discontinuous coefficients. Such equations, according to their general theory, have as a rule discontinuous solutions, while the mechanical content of the problem requires a sufficiently smooth solution of the resulting singular equations.

1. General Formulas of the Plane Theory of Elasticity. Fundamental Boundary Value Problems. Some Auxiliary Propositions

1°. To facilitate references, we present the well-known formulas of the plane theory of elasticity in the static case and give the formulation of the fundamental boundary value problems. The derivation of these formulas may be found in the book of N. I. Muskhelishvili [93] which will be referred to repeatedly in what follows.

By a plane elastic state is usually meant a plane deformation of an elastic medium parallel to a given plane (the deformation of a long cylinder with free bases) or its state of generalized plane stress (the deformation of a thin plate by forces lying in its plane). The determination of elastic equilibrium in these cases is reduced to seeking a solution of the biharmonic equation

$$\Delta\Delta u = f(x, y), \tag{1.1}$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

which satisfies certain boundary conditions. In an approximate consideration the problems of the equilibrium of a thin elastic plate subjected to normal loads are also reduced to the biharmonic equation. Plane problems and problems of the bending of thin plates are closely similar in the mathematical statement, as are their methods of solution. It appears advisable, therefore, to consider the two types of problem simultaneously, which we shall do in what follows.

The physical region, i.e., the region occupied by an elastic medium, will be denoted by S. We shall be concerned mostly with elastic media filling finite or infinite parts of the xy plane bounded by a single closed contour L.

The components of stress σ_x, σ_y, τ_{xy} and the components of displacement u, v in a system of Cartesian co-ordinates are expressed, in the absence of body forces, in terms of two analytic functions, φ and ψ, of one complex argument $z = x + iy$ by the well-known Kolosov-Muskhelishvili formulas

$$\left.\begin{aligned} \sigma_x + \sigma_y &= 2\,[\varphi'(z) + \overline{\varphi'(z)}], \\ \sigma_y - \sigma_x + 2i\tau_{xy} &= 2\,[\bar{z}\varphi''(z) + \psi'(z)], \end{aligned}\right\} \tag{1.2}$$

$$2\mu\,(u + iv) = \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}, \tag{1.3}$$

where μ, the shearing modulus, and $\varkappa$, Muskhelishvili's constant, are elastic constants. Also,

$$\varkappa = 3 - 4\nu \quad \text{in the case of plane strain,}$$

$$\varkappa = \frac{3-\nu}{1+\nu} \quad \text{in the case of a thin plate,}$$

where ν is Poisson's ratio.

Formulas (1.2) are based on the well-known Goursat formula giving the general complex representation of a stress function U satisfying (in the absence of body forces) the homogeneous biharmonic equation

$$2U = \bar{z}\varphi(z) + z\overline{\varphi(z)} + \chi(z) + \overline{\chi(z)} \qquad (\psi(z) = \chi'(z)). \tag{1.4}$$

From (1.4) it is easy to derive a very useful formula

$$\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y} = \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}. \tag{1.5}$$

Let AB be an arbitrary line imagined to be drawn within the body (in the region S), and $X_n\,ds$, $Y_n\,ds$ the components of the stress vector $\mathbf{F}_n\,ds$ applied to an element of arc ds of the profile AB on the right (on the positive side of the normal). We then have the equality

$$\int_A^B (X_n + iY_n)\,ds = -i\left[\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}\right]_A^B$$

which, together with (1.5), gives

$$\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} = i\int_A^B (X_n + iY_n)\,ds + \text{constant}. \tag{1.6}$$

From formulas (1.2) and (1.3) giving a general representation of the displacement and stress fields it follows that the problem of determining the elastic equilibrium of a body is equivalent to finding two analytic functions, $\varphi(z)$ and $\psi(z)$.

By the fundamental problems of the plane theory of elasticity are usually meant the following three.

The first fundamental problem consists in determining the elastic equilibrium of a body from external forces prescribed on its surface. The boundary condition for the functions φ and ψ is

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = f_1 + if_2 + \text{constant on } L, \tag{1.7}$$

where $f_1 + if_2$ is a given function of the points of the boundary L

$$f_1(s) + if_2(s) = i\int_0^s (X_n + iY_n)\,ds,$$

with t denoting the affix of a point on L and s its arc abscissa.

The second fundamental problem requires the determination of the elastic equilibrium of a body from given displacements of the points of its boundary. For determining the functions φ and ψ we have, according to formula (1.3), the boundary condition

$$\varkappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)} = 2\mu(g_1 + ig_2) \text{ on } L, \tag{1.8}$$

where $g_1(s)$ and $g_2(s)$ are given functions on L.

The fundamental mixed problem consists in determining the displacement and stress fields when external forces are prescribed over a part of the body boundary, and elastic displacements over the remainder. The boundary conditions here take the form of relations (1.7) and (1.8) assigned in the respective parts of the contour.

In applications one has often to deal also with the so-called *third problem* of the theory of elasticity when it is required to determine the elastic equilibrium of a body from given values, on its boundary, of the normal component v_n of the displacement vector and of the tangential component T of the external stress vector. The boundary conditions are prescribed in the form

$$v_n = f(t), \qquad T = h(t) \text{ on } L, \tag{1.9}$$

where f and h are given functions. This third problem has a definite mechanical significance when $h(t) = 0$ on L. It corresponds to contact (along the entire boundary L) between an elastic body and a rigid profile of given shape when the contact between the bodies takes place without friction.* On the basis of formulas (1.2) and (1.3) it is easy to write conditions (1.9) in complex form.

The functions φ and ψ are not entirely determined uniquely by the boundary conditions of the fundamental problems. For example, in the case of a finite region containing the origin of co-ordinates they may be subjected to the following additional conditions:

in the first problem

$$\varphi(0) = 0, \qquad \operatorname{Im} \varphi'(0) = 0, \qquad \psi(0) = 0; \tag{1.10}$$

in the second and mixed problems

$$\varphi(0) = 0 \quad \text{or} \quad \psi(0) = 0. \tag{1.11}$$

The functions φ and ψ are often referred to as the Kolosov-Muskhelishvili complex potentials.

In the case of a finite region S bounded by a single closed contour the functions φ and ψ are holomorphic in z. When the region is infinite and includes the point at infinity, the complex potentials have, in general, singularities at the point $z = \infty$ and admit, in its neighbourhood, of the representation

$$\left.\begin{aligned} \varphi(z) &= -\frac{X+iY}{2\pi(1+\varkappa)} \ln z + \Gamma z + \varphi_0(z), \\ \psi(z) &= \frac{\varkappa(X-iY)}{2\pi(1+\varkappa)} \ln z + \Gamma' z + \psi_0(z). \end{aligned}\right\} \tag{1.12}$$

Here $\varphi_0(z)$, $\psi_0(z)$ are holomorphic functions and X, Y, Γ, Γ' are constants. The first two of these constants are real and represent the projections on the x and y axes of the resultant of the external forces applied to the boundary of the region. The other two constants may, in general, have complex values and define a uniform field at infinity.

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* For applications, of course, of greater interest is the case when contact between the punch and the elastic body takes place along a part of the boundary of the latter. In this case the problem belongs to mixed ones.

In determining the elastic equilibrium (of an infinite medium of the kind considered here) the constants Γ and Γ' are always given. The quantities X and Y in the first problem are given in a natural way, and in the second and mixed problems they must be assigned in advance. Otherwise, the required elastic equilibrium will not be determined uniquely.

The formulas of the plane theory of elasticity often involve, along with the functions φ and ψ, other potentials, Φ and Ψ, which are the first z derivatives of the functions $\varphi(z)$ and $\psi(z)$

$$\Phi(z) = \varphi'(z), \qquad \Psi(z) = \psi'(z).$$

Below we shall sometimes use the term “plane problems”. This usually implies the first and second fundamental problems.

These *fundamental problems* for regions bounded by a single closed contour are entirely equivalent from a mathematical point of view. The difference between them is immaterial and lies in the following.* For finite regions, in the case of the first problem the right-hand sides of conditions (1.7) have to be subjected to restrictions which ensure the equilibrium of an elastic body, and in the case of the second problem there is no need for such restrictions. For infinite regions, as pointed out above, in the second problem one must assign, *in addition*, the components of the resultant of the external forces applied to the boundary, and in the first problem these components are given according to the *boundary conditions themselves.*

The term “plane problems” is therefore often replaced by the term “plane problem” meaning by this the first fundamental problem.

2°. The most efficient methods of solving plane problems, developed in the work of N. I. Muskhelishvili and his followers, are based on the application of the techniques of the theory of functions of a complex variable and make an essential use of the conformal mapping of a given physical region onto a unit circle in the plane of an auxiliary variable. Below are presented the basic relations in curvilinear co-ordinates associated with a given mapping

$$z = \omega(\zeta). \tag{1.13}$$

---

* In the case of multiply connected bodies the difference is also inessential from the point of view of the general theory of boundary value problems.

Instead of $\varphi(z)$, $\psi(z)$ we write $\varphi_1(z)$, $\psi_1(z)$ and introduce the notation

$$\varphi(\zeta)=\varphi_1(z)=\varphi_1[\omega(\zeta)], \qquad \psi(\zeta)=\psi_1(z)=\psi_1[\omega(\zeta)],$$
$$\Phi(\zeta)=\frac{d\varphi_1}{dz}=\frac{\varphi'(\zeta)}{\omega'(\zeta)}, \qquad \Psi(\zeta)=\frac{d\psi_1}{dz}=\frac{\psi'(\zeta)}{\omega'(\zeta)}.$$

In terms of these new functions φ and ψ the boundary conditions of the fundamental problems become

$$\varphi(\sigma)+\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\varphi'(\sigma)}+\overline{\psi(\sigma)}=f_1+if_2+\text{constant on } \gamma \tag{1.14}$$

for the first problem, and

$$\varkappa\varphi(\sigma)-\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\varphi'(\sigma)}-\overline{\psi(\sigma)}=2\mu(g_1+ig_2) \text{ on } \gamma \tag{1.15}$$

for the second problem. In the above equalities γ denotes the contour of a unit circle in the plane of the variable ζ, and σ the affix of a point on it; $\sigma=e^{i\vartheta}$, where ϑ is the polar angle.

Denote, further, by σ_ρ, σ_ϑ, $\tau_{\rho\vartheta}$ and v_ρ, v_ϑ, respectively, the stress and displacement components referred to our curvilinear co-ordinates. There are the following complex representation formulas:

$$\left.\begin{aligned}\sigma_\rho+\sigma_\vartheta&=2[\Phi(\zeta)+\overline{\Phi(\zeta)}],\\ \sigma_\vartheta-\sigma_\rho+2i\tau_{\rho\vartheta}&=\frac{2\zeta^2}{\rho^2\omega'(\zeta)}[\overline{\omega(\zeta)}\Phi'(\zeta)+\omega'(\zeta)\Psi(\zeta)],\end{aligned}\right\} \tag{1.16}$$

$$2\mu|\omega'(\zeta)|(v_\rho+iv_\vartheta)=\frac{\bar{\zeta}}{\rho}\overline{\omega'(\zeta)}\left\{\varkappa\varphi(\zeta)-\frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\varphi'(\zeta)}-\overline{\psi(\zeta)}\right\}, \tag{1.17}$$

where $\zeta=\rho e^{i\vartheta}$.

We present one more formula resulting from (1.16):

$$\sigma_\rho-i\tau_{\rho\vartheta}=\Phi(\zeta)+\overline{\Phi(\zeta)}-\frac{\zeta^2}{\rho^2\omega'(\zeta)}\{\overline{\omega(\zeta)}\Phi'(\zeta)+\omega'(\zeta)\Psi(\zeta)\}. \tag{1.18}$$

3°. In the approximate theory of the bending of a plate by normal forces distributed over its surface, the biharmonic equation (1.1) is satisfied by the deflections of the points of the middle surface. For these deflections $w(x, y)$ we have the equation

$$\Delta\Delta w=\frac{q}{D}, \tag{1.19}$$

where $q(x, y)$ is the load intensity and D is the flexural rigidity.

After finding a particular solution of Eq. (1.19), we can use the Goursat formula (1.4) as before and represent the general solution of this equation in complex form. Ultimately all desired quantities characterizing the equilibrium of the bent plate can be expressed in terms of the Kolosov-Muskhelishvili potentials and the particular solution of Eq. (1.19). The complex representation formulas for this case are similar to formulas (1.2) and are of the form

$$\left.\begin{aligned} G_y - G_x + 2iH_{xy} &= 4D(1-\nu)\,[\bar{z}\varphi''(z) + \psi'(z)] + G_y^{(0)} - G_x^{(0)} + 2iH_{xy}^{(0)},\\ G_x + G_y &= -4D(1+\nu)\,[\varphi'(z) + \overline{\varphi'(z)}] + G_x^{(0)} + G_y^{(0)},\\ N_x - iN_y &= -8D\varphi''(z) + N_x^{(0)} - iN_y^{(0)}, \end{aligned}\right\} \tag{1.20}$$

$$\left.\begin{aligned} w &= u + w_0,\\ u &= \bar{z}\varphi(z) + z\overline{\varphi(z)} + \chi(z) + \overline{\chi(z)}; \end{aligned}\right\} \tag{1.21}$$

here $\psi(z) = \chi'(z)$; G_x, G_y are the bending moments, H_{xy} is the twisting moment, N_x, N_y are the shearing forces per unit length; $G_x^{(0)}$, . . ., $N_y^{(0)}$ are the same quantities pertaining to the chosen particular solution w_0 of Eq. (1.19).

The derivation of the preceding formulas and the investigation of the general properties of the potentials φ and ψ for bending problems were given by S. G. Lekhnitskii [73]. The degree of determinateness of the potentials $\varphi(z)$ and $\psi(z)$ is the same as in the plane case. More precisely, with the state of stress given, i.e., with G_x, G_y, H_{xy} given, they may be subjected to the additional conditions (1.10).

In order to determine the deflections $w(x, y)$, it is necessary to add to Eq. (1.19) boundary conditions corresponding to a particular mode of fixing the boundary.

Three main problems are considered.

I. *The edge of a plate is clamped.* This means that on the boundary of the region S occupied by the middle surface of the plate the following conditions must be fulfilled:

$$w = 0, \qquad \frac{dw}{dn} = 0 \qquad (\text{on } L) \tag{1.22}$$

where n is the outward normal to the contour.

II. *The edge of a plate is free.* The boundary conditions are

$$\left.\begin{aligned} G(w) &\equiv \nu\Delta w + (1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 w}{\partial x\,\partial y}\cos\theta\sin\theta + \right.\\ &\qquad\left. + \frac{\partial^2 w}{\partial y^2}\sin^2\theta\right] = 0,\\ H(w) &\equiv \frac{d\,\Delta w}{dn} + (1-\nu)\frac{d}{ds}\left[\frac{\partial^2 w}{\partial x\,\partial y}\cos 2\theta + \right.\\ &\qquad\left. + \left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2}\right)\cos\theta\sin\theta\right] = 0 \quad (\text{on } L), \end{aligned}\right\} \tag{1.23}$$

where θ is the angle made with the x axis by the outward normal to the contour. The expressions for G and H multiplied by $-D$ represent, respectively, the bending moment and the generalized shearing force per unit length acting on a linear element of the middle plane with normal n.

III. *The edge of a plate is simply supported.* To simply supported edges correspond the boundary conditions

$$w = 0, \qquad G(w) = 0 \qquad (\text{on } L). \tag{1.24}$$

Besides these main types of boundary condition there frequently occur mixed conditions, which are of particular interest for applications, when, for example, a part of the plate edge is clamped, the remainder is free, and so on.

If, in addition to the load q distributed over the middle surface of the plate, it is acted on by bending moments and normal forces applied along the edge, the bending equation (1.19) remains the same and the boundary conditions (1.22) to (1.24) become non-homogeneous. The left-hand sides of these equalities represent some given functions of the arc of the contour L (S. G. Lekhnitskii [73]).

The values of w and $\frac{dw}{dn}$ on the contour L completely determine the boundary values of the first derivatives of the deflections w on the same contour. Problem I (the edge of a plate is clamped), commonly referred to as the fundamental biharmonic problem, is therefore mathematically equivalent to the first fundamental problem of the plane theory of elasticity.* The boundary conditions (1.22) for the regions (bounded by a single closed contour) we are considering are exactly the same as condition (1.7).

Let $\varphi(z)$, $\chi(z)$ be arbitrary holomorphic functions in the region S, w a regular biharmonic function ($w_0 \equiv 0$) corresponding to them by formula (1.21), G and H the differential operators, defined by formula (1.23), formed for the function w. Then the following formula of S. G. Lekhnitskii [73] and I. N. Vekua [136] holds on any smooth part of the boundary of the region:

$$G(w) + i\int^{s} H(w)\,ds + iC = $$

$$= -2(1-\nu)\frac{d}{dt}\left[-\varkappa\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)}\right]. \tag{1.25}$$

Here $\varphi(t)$, $\varphi'(t)$, $\psi(t)$, as always, denote the boundary values of the corresponding functions, C is a constant of integration and

$$\varkappa = \frac{3+\nu}{1-\nu}. \tag{1.26}$$

According to this formula, the boundary conditions of Problem II in complex variable notation can be put into the form of the boundary conditions (1.8) of the second fundamental plane problem. The only difference is that in the case of the problem involving free edges the right-hand side of condition (1.8) is prescribed to within an additive term of the form $iCt + C_1$, where C and C_1 are (respectively real and complex) arbitrary constants. In the case of a single contour, however, these constants may always be neglected due to the arbitrariness in the selection of the potentials φ and ψ.

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* For further details see N. I. Muskhelishvili [93], Secs. 40, 41.

Finally, conditions (1.24) for simply supported edges can be written in terms of the functions φ and ψ in the form (see, for example, the author's paper [38])

$$\left.\begin{aligned} &\operatorname{Re}\left\{\lambda\varphi'(t) - \left(\frac{dt}{ds}\right)^2 [\bar{t}\varphi''(t) + \psi'(t)]\right\} = h(t), \\ &\operatorname{Re}\left\{\frac{d\bar{t}}{ds}\,[\overline{\varphi(t)} + \bar{t}\varphi'(t) + \psi(t)]\right\} = g(t) \quad (\text{on } L), \end{aligned}\right\} \tag{1.27}$$

where h and g are given functions on L, and the constant λ is

$$\lambda = \frac{2(1+\nu)}{1-\nu}\,.$$

It can easily be verified that the problem defined by conditions (1.27) is mathematically equivalent to the third problem of the plane theory of elasticity.

The external resemblance between plane problems and problems of the bending of thin plates enables one to apply the known methods of solution for some problems to the solution of the others. This was first illustrated in the work of A. I. Lur'e [75].

4°. Let L be a simple, closed or unclosed, smooth line (curve) in a finite part of the z plane, and $f(t)$ some function, in general complex, given on L and integrable in the ordinary sense. Consider the function

$$F(z) = \frac{1}{2\pi i}\int_L \frac{f(t)\,dt}{t-z}, \tag{1.28}$$

where z is some point of the plane. It is easily verified that the function $F(z)$ is holomorphic in any finite part of the plane, not containing points of L, and is regular at infinity. Integral (1.28) is commonly termed a Cauchy-type integral along the line L.

The Cauchy-type integral possesses many remarkable properties of which we shall indicate only one.

Let the function $f(t)$ called the density function of the Cauchy-type integral satisfy a Hölder condition on L. This means that for any two points, t_1 and t_2, of the line L the following inequality is fulfilled:

$$|f(t_2) - f(t_1)| \leqslant A\,|t_2 - t_1|^{\alpha}, \tag{1.29}$$

where A and α are some constants (independent of the choice of points t_1, t_2), with $0 < \alpha \leqslant 1$.

Then the function $F(z)$ is continuously extendible to the line L from the left and right of it (in relation to the chosen positive sense). The following Sokhotskii-Plemelj formulas* hold for the limiting values of the function from the left and from the right, designated by the symbols $F^+(t)$ and $F^-(t)$ respectively:

$$\left.\begin{aligned} F^+(t_0) &= \frac{1}{2} f(t_0) + \frac{1}{2\pi i} \int\limits_L \frac{f(t)\,dt}{t-t_0}, \\ F^-(t_0) &= -\frac{1}{2} f(t_0) + \frac{1}{2\pi i} \int\limits_L \frac{f(t)\,dt}{t-t_0} \quad (t_0 \text{ on } L), \end{aligned}\right\} \tag{1.30}$$

where the integral on the right-hand sides is understood as a Cauchy principal value.

In the case when L is an unclosed arc formulas (1.30) are valid everywhere on L except at the ends; when L is a closed contour, however, these formulas are valid at any point of the contour without exception.

The definition of the Cauchy-type integral may be generalized in various directions. It is possible, for example, to consider this integral in more general cases when L is a finite (and even infinite) set of separately lying closed and finite unclosed curves, when the function $f(t)$ belongs to a more general class than the class of Hölder continuous functions, or, say, when an unclosed line extends to infinity with one or both of its ends.

Detailed information about Cauchy-type integrals may be found in the monographs by I. I. Privalov [105] and by N. I. Muskhelishvili [92].

2. Various Representations of Complex Potentials. Integral Equations of the Plane Problem

The section is an introductory one. Its main object is the interpretation of the well-known representations of the elastic potentials by Cauchy-type integrals. In passing, it may be useful to indicate some other forms of the same potentials also suitable for the solution of boundary value problems. The potentials in question will play a significant part in the study of mixed problems considered in the next two sections.

We begin our presentation with a brief (and far from complete) review of the results on the general methods of solution of the plane problem.

1°. **Basic integral equations.** For ease of description we restrict our consideration to regions S in the plane $z = x + iy$ which are bounded by a single simple closed contour L. The finite and infinite regions with the boundary L are denoted, respectively, by S^+ and S^-. Suppose that the origin lies inside S^+, the positive sense on L leaves the region S^+ on the left, and the angle made with some fixed direction by the tangent to L satisfies the Hölder condition along the arc of the contour. For definiteness, we shall consider the case of finite regions.

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* The proofs of these formulas will be found in various textbooks on complex function theory.

The first and second fundamental problems will be treated simultaneously. According to (1.7) and (1.8), the boundary conditions for both problems are of the form

$$k\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = f(t) \quad \text{on} \quad L, \tag{2.1}$$

where k is a constant and $f(t)$ is a given function on L; in the case of the first fundamental problem $k = 1$,

$$f(t) = i\int_0^s (X_n + iY_n)\,ds, \tag{2.2}$$

while for the second fundamental problem $k = -\varkappa$,

$$f(t) = -2\mu\,(g_1 + ig_2). \tag{2.3}$$

By $\varphi(t)$, $\varphi'(t)$, $\psi(t)$ are meant the boundary values on L of the corresponding functions which exist, and are consequently continuous, by hypothesis. A solution $\varphi(z)$, $\psi(z)$ possessing the above property is commonly termed a regular solution of the plane problem.

Condition (2.1) is now rewritten in equivalent form

$$\psi(t) = -k\overline{\varphi(t)} - \bar{t}\varphi'(t) + \overline{f(t)} \quad \text{on} \quad L \tag{2.1'}$$

and the following notation is introduced:

$$\Omega(z, \bar{z}) = k\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}. \tag{2.4}$$

It is easily verified that the regular solution of problem (2.1), if it exists, is representable in the form (N. P. Vekua [140])

$$\left.\begin{aligned} \varphi(z) &= \frac{1}{2\pi i}\int_L \frac{\omega(t)\,dt}{t-z}, \\ \psi(z) &= -\frac{1}{2\pi i}\int_L \frac{k\overline{\omega(t)} + \bar{t}\omega'(t)}{t-z}\,dt + A(z), \end{aligned}\right\} \tag{2.5}$$

$$A(z) = \frac{1}{2\pi i}\int_L \frac{\overline{f(t)}\,dt}{t-z} \qquad (z \text{ in } S^+), \tag{2.6}$$

where $\omega(t)$ is some function of the point on the contour L which is to be determined from the boundary condition.

For the right-hand sides of formulas (2.5)

$$\Omega(z, \bar{z}) = \frac{k}{2\pi i} \int_L \omega(t) \left[\frac{dt}{t-z} + \frac{\overline{dt}}{\bar{t}-\bar{z}} \right] + \frac{1}{2\pi i} \int_L \frac{t-z}{\bar{t}-\bar{z}} \overline{\omega'(t)}\, \overline{dt} + \overline{A(z)}.$$

By allowing the point z to tend from the inside of S^+ to some point t_0 of the contour L and using the Sokhotskii-Plemelj formulas (1.30), we obtain, by (2.1),

$$\frac{k}{\pi i} \int_L \frac{\omega(t)\, dt}{t-t_0} - \frac{k}{2\pi i} \int_L \omega(t)\, d \ln \frac{t-t_0}{\bar{t}-\bar{t}_0} - \frac{1}{2\pi i} \int_L \overline{\omega(t)}\, d \frac{t-t_0}{\bar{t}-\bar{t}_0} =$$

$$= f(t_0) - \overline{A^+(t_0)}. \tag{2.7}$$

This is precisely the singular equation of the problem indicated in the above-cited work of N. P. Vekua. The question of its solvability is of no interest to us.

In representation (2.5) the function $\omega(t)$ is not subject to any restriction apart from being smooth, which is necessary for the regularity of the required solution. If we put, in particular, $\omega(t) = \varphi(t)$, then (2.5) will merely express Cauchy's integral formula for $\varphi(z)$ and $\psi(z)$, and Eq. (2.7), on the basis of the obvious equality

$$\frac{1}{\pi i} \int_L \frac{\varphi(t)\, dt}{t-t_0} = \varphi(t_0)$$

will transform into a Fredholm equation in the boundary values of the potential $\varphi(z)$:

$$k\varphi(t_0) - \frac{k}{2\pi i} \int_L \varphi(t)\, d \ln \frac{t-t_0}{\bar{t}-\bar{t}_0} - \frac{1}{2\pi i} \int_L \overline{\varphi(t)}\, d \frac{t-t_0}{\bar{t}-\bar{t}_0} = -\overline{A^-(t_0)}. \tag{2.8}$$

Equation (2.8) was indicated by N. I. Muskhelishvili [89, 90] in 1934. Its original derivation given in N. I. Muskhelishvili's note [89] was not based on any representations of the form of (2.5) and started only from the well-known Cauchy theorem on holomorphic functions. Representation (2.5) with $\omega(t) = \varphi(t)$, i.e., Cauchy's formula, simplifies the derivation of the equation and somewhat weakens the restrictions imposed *a priori* on the required solution.

The Muskhelishvili integral equation attracted the attention of many specialists who devoted a number of interesting works to various aspects of its application. Detailed information on the investigations in this direction may be found in the monograph by N. I. Muskhelishvili [93], Sec. 98.

It is well known that the homogeneous equation (2.8) with $k = 1$ admits a solution $\varphi_0(t)$ involving three arbitrary real constants

$$\varphi_0(t) = i\varepsilon t + \alpha + i\beta. \tag{2.9}$$

It is proved that two of the orthogonality conditions necessary for the solvability of (2.8) are automatically satisfied because of the special form of the right-hand side of the equation, and the third condition expresses zero resultant couple of the external forces applied to the boundary of an elastic body. In the case of the second fundamental problem there is no constant ε in (2.9) and the conditions for solvability are automatically fulfilled. It appears from the above that if (in the case of the first problem) the conditions of the statics of rigid bodies are fulfilled, Eq. (2.8) is always solvable and its solution $\varphi(t)$, subject to certain conditions for smoothness, gives a regular solution of the plane problem.

D. I. Sherman [121, 122] has suggested the following integral epresentation for the complex potentials:

$$\left.\begin{aligned} \varphi(z) &= \frac{1}{2\pi i}\int_L \frac{\omega(t)\,dt}{t-z}, \\ \psi(z) &= \frac{k}{2\pi i}\int_L \frac{\overline{\omega(t)}\,dt}{t-z} + \frac{1}{2\pi i}\int_L \frac{\omega(t)\,\overline{dt}}{t-z} - \frac{1}{2\pi i}\int_L \frac{\bar{t}\omega(t)\,dt}{(t-z)^2} \quad (z \text{ in } S^+), \end{aligned}\right\} \tag{2.10}$$

where $\omega(t)$ is, as before, some function of the point on L.

According to these formulas, for expression (2.4) we have

$$\Omega(z, \bar{z}) = \frac{k}{2\pi i}\int_L \omega(t)\left[\frac{dt}{t-z} - \frac{\overline{dt}}{\bar{t}-\bar{z}}\right] + \frac{1}{2\pi i}\int_L \frac{t-z}{\bar{t}-\bar{z}}\,\overline{\omega'(t)}\,\overline{dt}.$$

From this, as above, by passing to the limit as $z \to t_0$, we obtain, by the boundary condition (2.1), the well-known Sherman-Lauricella equation

$$k\omega(t_0) + \frac{k}{2\pi i}\int_L \omega(t)\,d\ln\frac{t-t_0}{\bar{t}-\bar{t}_0} - \frac{1}{2\pi i}\int_L \overline{\omega(t)}\,d\frac{t-t_0}{\bar{t}-\bar{t}_0} = f(t_0). \tag{2.11}$$

In the case of the first problem, when $k = 1$, to the left-hand side of the preceding equation is added the term

$$b\left(\frac{1}{t_0} - \frac{1}{\bar{t}_0} + \frac{t_0}{\bar{t}_0^2}\right), \qquad b = \frac{1}{2\pi i}\int_L \left\{\frac{\omega(t)}{t^2}\,dt + \frac{\overline{\omega(t)}}{\bar{t}^2}\,\overline{dt}\right\}$$

and it is proved that, if the condition for zero resultant couple of the external forces is fulfilled, the modified equation has no solutions different from those of the original equation, and the corresponding homogeneous equation has no non-trivial solutions. In the case of the second problem ($k = -\varkappa$) Eq. (2.11) need not be corrected. In other words, the Sherman-Lauricella equation always has a unique solution which, subject to certain additional conditions, leads to a regular solution of the plane problem.

In the derivation of all the foregoing equations it has been assumed that the functions $\omega(t)$ and $f(t)$ are Hölder continuous on L. This is ultimately inadequate, however. The condition for regularity of $\varphi(z)$, $\psi(z)$ requires the solutions of integral equations to have first-order derivatives with respect to t (or s) satisfying the Hölder condition. The existence of such solutions may be guaranteed on the assumption that the curvature of the contour L and the derivative $f'(t)$ satisfy the Hölder condition on L. The above restrictions will be assumed fulfilled.

Revert to Eq. (2.11). This equation, as well as representation (2.10) leading to it, possesses several advantages over other known equations, which is discussed in detail in N. I. Muskhelishvili's book [93] (see Secs. 101, 102).

The main advantage of Sherman's method lies in the fact that it reduces the fundamental problems to equations very convenient for investigations and applies equally well to the solution of some other problems, including mixed ones. This method is still widely used for general investigations of boundary value problems and not infrequently serves the purpose of constructing their effective solution. Sherman's potentials (2.10) will be used in the present chapter for studying mixed problems.

D. I. Sherman did not indicate the ways of obtaining his formulas. According to the interpretation given by N. I. Muskhelishvili [93], Sec. 101, they can be derived by using the following idea due to Fredholm.

We solve problem (2.1) for the upper half-plane under usual assumptions. As is known, the solution is represented in closed form and is identical with formulas (2.10) if we set in them

$$\omega(t)=\frac{1}{k}f(t)$$

and if we mean by L the whole real axis $y=0$. Consequently, the solution for the half-plane $\varphi(z)$, $\psi(z)$, if in the Cauchy-type integrals defining it the given function $f(t)$ on the axis is formally replaced by the unknown density function $\omega(t)$ and the line of integration by the boundary of the region S^+ under consideration, gives an integral representation of the complex potentials which reduces the plane problem for S^+ to a Fredholm equation.

Below is given a method of constructing these potentials from physical considerations.

2°. **Construction of elastic potentials.** Let a concentrated force with components $p(t)$, $q(t)$ be applied at some point t of the plane $z=x+iy$. The action of the concentrated force produces a stress field in the entire plane defined at any point z different from t by the formulas (N. I. Muskhelishvili [93], Sec. 56a)

$$\left.\begin{aligned}\varphi(z;\,t)&=\frac{\omega(t)}{2\pi i}\ln(z-t)-\frac{\omega(t)}{2\pi i}\ln t,\\ \psi(z;\,t)&=\frac{\varkappa\overline{\omega(t)}}{2\pi i}\ln(z-t)-\frac{\omega(t)}{2\pi i}\,\frac{\bar{t}}{z-t}-\frac{\varkappa\overline{\omega(t)}}{2\pi i}\ln t,\end{aligned}\right\}\tag{2.12}$$

where, as will be recalled, $\varkappa$ is Muskhelishvili's constant and

$$\omega(t)=\frac{q(t)-ip(t)}{1+\varkappa}.\tag{2.13}$$

The last terms on the right-hand sides of (2.12) have been introduced for convenience, the significance of which will be apparent from what follows. These terms are independent of z and so will affect only the rigid-body displacement of the plane as a whole.

We place the point t on the contour L, multiply both sides of equalities (2.12) by ds and integrate over L. As a result, we obtain functions of z holomorphic in the region S^+:

$$\left.\begin{aligned}\varphi(z)&=\frac{1}{2\pi i}\int_L\omega(t)\ln\left(\frac{z}{t}-1\right)ds,\\ \psi(z)&=\frac{\varkappa}{2\pi i}\int_L\overline{\omega(t)}\ln\left(\frac{z}{t}-1\right)ds+\frac{1}{2\pi i}\int_L\frac{\bar{t}\omega(t)\,ds}{t-z}.\end{aligned}\right\}\tag{2.14}$$

They describe the stress field produced within S^+ by the distributed, along L, load $(p,\,q)$ applied at a point of the contour to the infinite body.

Instead of $\varkappa$ in front of the integral for $\psi(z)$, we take a numerical parameter k which may assume two values, $k=1,\ -\varkappa$. We obtain a pair of complex potentials:

$$\left.\begin{aligned}\varphi(z)&=\frac{1}{2\pi i}\int_L\omega(t)\ln\left(\frac{z}{t}-1\right)ds,\\ \psi(z)&=\frac{k}{2\pi i}\int_L\overline{\omega(t)}\ln\left(\frac{z}{t}-1\right)ds+\frac{1}{2\pi i}\int_L\frac{\bar{t}\omega(t)\,ds}{t-z}\\ &\qquad\qquad(k=1,\ -\varkappa).\end{aligned}\right\}\tag{2.15}$$

It should be observed that when $k=1$ the function $\omega(t)$ in the above formulas has a definite physical significance. Namely, the expression $i(1+\varkappa)\,\omega(t)$ represents a force, vector $(p,\,q)$, applied at the point t to an unbounded body of incompressible material in a state of plane strain. When $k=-\varkappa$, it has no physical significance any more since the constant $\varkappa$, according to its definition, cannot take negative values. In this case $\omega(t)$ must be understood as a fictitious load.

In the second of formulas (2.15) we replace k by $-k$ and retain the same values for the parameter k. We then obtain another pair of potentials:

$$\left.\begin{aligned}
\varphi(z) &= \frac{1}{2\pi i}\int_L \omega(t)\ln\left(\frac{z}{t}-1\right)ds,\\
\psi(z) &= -\frac{k}{2\pi i}\int_L \overline{\omega(t)}\ln\left(\frac{z}{t}-1\right)ds+\frac{1}{2\pi i}\int_L \frac{\bar{t}\omega(t)\,ds}{t-z}\\
&\qquad\qquad (k=1,\ -\varkappa).
\end{aligned}\right\}\qquad(2.16)$$

The density function $\omega(t)$ in these formulas is not related to a real load for $k=1$, but it has a physical meaning for $k=-\varkappa$ when the expression $i(1+\varkappa)\,\omega(t)$ is interpreted as a force applied at a point of a plane of arbitrary elastic material.

From formulas (2.15) and (2.16) it is possible to obtain some other potentials, less smooth in the closed region. One may, for example, consider holomorphic functions $\varphi(z)$, $\psi(z)$ in S obtainable from the above by differentiating the kernels of the corresponding representations with respect to the arc length s. In this way we introduce into consideration the following two new pairs of complex potentials:

$$\left.\begin{aligned}
\varphi(z) &= \frac{1}{2\pi i}\int_L \omega(t)\,d\ln\left(\frac{z}{t}-1\right),\\
\psi(z) &= \frac{k}{2\pi i}\int_L \overline{\omega(t)}\,d\ln\left(\frac{z}{t}-1\right)+\frac{1}{2\pi i}\int_L \omega(t)\,d\left\{\frac{\bar{t}}{t-z}\right\}\\
&\qquad\qquad (k=1,\ -\varkappa);
\end{aligned}\right\}\qquad(2.17)$$

$$\left.\begin{aligned}
\varphi(z) &= \frac{1}{2\pi i}\int_L \omega(t)\,d\ln\left(\frac{z}{t}-1\right),\\
\psi(z) &= -\frac{k}{2\pi i}\int_L \overline{\omega(t)}\,d\ln\left(\frac{z}{t}-1\right)+\frac{1}{2\pi i}\int_L \omega(t)\,d\left\{\frac{\bar{t}}{t-z}\right\}\\
&\qquad\qquad (k=1,\ -\varkappa).
\end{aligned}\right\}\qquad(2.18)$$

The function $\omega(t)$ in the above formulas has a different meaning, however. By analogy with harmonic potential theory we can relate this function to a "force dipole" (two large, equal and opposite, concentrated forces a small distance apart) and attach to it a definite physical meaning for $k=1$ in formula (2.17) (for incompressible material) and for $k=-\varkappa$ in formula (2.18). In these cases the quantity $i(1+\varkappa)\,\omega(t)$ may naturally be called the moment of a dipole placed at the point t of the unbounded plane when the direction of the "dipole axis"* coincides in the limit with the positive direction of the tangent to L at the same point. In the above cases functions (2.17) and (2.18) define the stress field produced within S^+ by force dipoles distributed along the contour L.

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* The segment connecting the points of application of opposite concentrated forces.

It may be recalled that the additive constants in the functions φ and ψ can always be omitted. Hence, *formulas* (2.17) *are nothing other than D. I. Sherman's representation* (2.10), *and formulas* (2.18) *are identical, apart from the term* $A(z)$ *defined by equality* (2.6), *with N. P. Vekua's representation* (2.5). The difference in the second case is insignificant as the function $A(z)$ is specified by the problem itself and will, of course, affect only the right-hand side of the integral equation.

It has already been said about the application of potentials (2.17) and (2.18) to the solution of the plane problem. Let us now show that (2.15) and (2.16) also solve the plane problem, representation (2.15) leading to a Fredholm equation, and representation (2.16) to a singular integral equation of the first kind.

3°. **Some other integral equations of the plane problem.** Suppose that the function $\omega(t)$ in formulas (2.15), (2.16) satisfies the Hölder condition on L, and the contour L satisfies the condition stated at the beginning of Par. 1°.

We begin with representation (2.15) by means of which we set up expression (2.4). We have

$$\Omega(z, \bar{z}) = \frac{k}{2\pi i}\int\limits_L \omega(t)\Big[\ln\frac{z}{t} - 1\Big) - \overline{\ln\Big(\frac{z}{t} - 1\Big)}\Big] - \frac{1}{2\pi i}\int\limits_L \frac{t-z}{\bar{t}-\bar{z}}\overline{\omega(t)}\,ds.$$

The right-hand side of the above equality is continuous in the closed region and the boundary condition of the problem gives

$$\frac{k}{2\pi i}\int\limits_L \omega(t)\Big[\ln\Big(\frac{t_0}{t} - 1\Big) - \overline{\ln\Big(\frac{t_0}{t} - 1\Big)}\Big] - \\ - \frac{1}{2\pi i}\int\limits_L \frac{t-t_0}{\bar{t}-\bar{t}_0}\overline{\omega(t)}\,ds = f(t_0). \qquad (2.19)$$

With our assumptions about $\omega(t)$ and L it is not difficult to prove the validity of the following differentiation formulas:

$$\left.\begin{aligned} &\frac{d}{ds_0}\frac{1}{\pi i}\int\limits_L \omega(t)\ln\Big(\frac{t_0}{t} - 1\Big)\,ds = -\omega(t_0) + \\ &\qquad + \frac{1}{\pi i}\int\limits_L \omega(t)\frac{\partial}{\partial s_0}\Big\{\ln\Big(\frac{t_0}{t} - 1\Big)\Big\}\,ds, \\ &\frac{d}{ds_0}\frac{1}{\pi i}\int\limits_L \omega(t)\,\overline{\ln\Big(\frac{t_0}{t} - 1\Big)}\,ds = \omega(t_0) + \\ &\qquad + \frac{1}{\pi i}\int\limits_L \omega(t)\frac{\partial}{\partial s_0}\Big\{\overline{\ln\Big(\frac{t_0}{t} - 1\Big)}\Big\}\,ds. \end{aligned}\right\} \qquad (2.20)$$

By differentiating both sides of equality (2.19) with respect to the arc length s_0 and using (2.20), we arrive at the Fredholm equation

$$k\omega(t_0)-\frac{k}{2\pi i}\int\limits_L \frac{\partial}{\partial s_0}\left\{\ln\frac{t-t_0}{\bar{t}-\bar{t}_0}\right\}\omega(t)\,ds+$$

$$+\frac{1}{2\pi i}\int\limits_L \frac{\partial}{\partial s_0}\left\{\frac{t-t_0}{\bar{t}-\bar{t}_0}\right\}\overline{\omega(t)}\,ds=F(t_0), \tag{2.21}$$

where in the case of the first problem

$$F = Y_n - iX_n \tag{2.22}$$

and for the second

$$F=2\mu\left(\frac{dg_1}{ds}+i\frac{dg_2}{ds}\right). \tag{2.23}$$

Here X_n, Y_n are the components of the external force vector, g_1, g_2 are the elastic displacements of the points of the contour.

The analysis of Eq. (2.21) presents no difficulty. Indeed, the associated homogeneous equation for (2.21)

$$k\nu(t_0)-\frac{k}{2\pi i}\int\limits_L \nu(t)\,d\ln\frac{t-t_0}{\bar{t}-\bar{t}_0}+\frac{1}{2\pi i}\int\limits_L \overline{\nu(t)}\,d\,\frac{t-t_0}{\bar{t}-\bar{t}_0}=0, \tag{2.24}$$

upon substitution of $\nu(t) = i\omega(t)$, transforms into the (homogeneous) equation (2.8) all solutions of which are determined by (2.9). It follows that the general solution of (2.24) is given by the formula

$$\nu(t) = \varepsilon t + \alpha + i\beta, \tag{2.25}$$

where ε, α, β are arbitrary real constants; when $k = -\varkappa$, we have $\varepsilon = 0$.

On the basis of (2.25) and the familiar proposition of the theory of singular integral equations the conditions for solvability of (2.21) are represented as

$$\int\limits_L F(t)\,ds=0, \qquad \operatorname{Re}\int\limits_L \overline{F(t)}\,t\,ds=0. \tag{2.26}$$

The preceding equalities in the case of the first fundamental problem express, according to (2.22), the well-known conditions in the statics of rigid bodies which will be assumed fulfilled. In the case of the second fundamental problem, however, the second of conditions (2.26) is absent, and the first is automatically satisfied because of (2.23). Consequently, Eq. (2.21) is always solvable and its solution, provided the right-hand side $F(t)$ satisfies the Hölder condition on L, determines a regular solution of the plane problem. The solution of (2.21) will involve three (when $k = 1$) or two (when $k = -\varkappa$) arbitrary real constants.

We now turn to representation (2.16). This time

$$\Omega(z, \bar{z}) = \frac{k}{2\pi i} \int_L \omega(t) \left[\ln\left(\frac{z}{t} - 1\right) + \overline{\ln\left(\frac{z}{t} - 1\right)} \right] - \\ - \frac{1}{2\pi i} \int_L \frac{t - z}{\bar{t} - \bar{z}} \overline{\omega(t)}\, ds$$

and the boundary condition becomes

$$\frac{k}{\pi i} \int_L \omega(t) \ln \left| \frac{t_0}{t} - 1 \right| ds - \frac{1}{2\pi i} \int_L \frac{t - t_0}{\bar{t} - \bar{t}_0} \overline{\omega(t)}\, ds = f(t_0). \qquad (2.27)$$

By differentiating the equality with respect to s_0 and noting that

$$\frac{\partial}{\partial s_0} \ln |t - t_0| = -\frac{t_0'}{t - t_0} - \frac{1}{2} \frac{\partial}{\partial s_0} \ln \frac{t - t_0}{\bar{t} - \bar{t}_0} \qquad \left(t' = \frac{dt}{ds}\right),$$

we obtain the singular integral equation of the first kind

$$\frac{k}{\pi i} \int_L \frac{t_0' \omega(t)\, ds}{t - t_0} + \frac{k}{2\pi i} \int_L \frac{\partial}{\partial s_0} \left\{ \ln \frac{t - t_0}{\bar{t} - \bar{t}_0} \right\} \omega(t)\, ds + \\ + \frac{1}{2\pi i} \int_L \frac{\partial}{\partial s_0} \left\{ \frac{t - t_0}{\bar{t} - \bar{t}_0} \right\} \overline{\omega(t)}\, ds = F(t_0). \qquad (2.28)$$

The right-hand side $F(t)$ is, as before, of the form of (2.22) and (2.23).

The associated homogeneous equation for (2.28)

$$\frac{k}{\pi i} \int_L \frac{\nu(t)\, dt}{t - t_0} - \frac{k}{2\pi i} \int_L \nu(t)\, d \ln \frac{t - t_0}{\bar{t} - \bar{t}_0} - \frac{1}{2\pi i} \int_L \overline{\nu(t)}\, d\, \frac{t - t_0}{\bar{t} - \bar{t}_0} = 0 \qquad (2.29)$$

is exactly the same as Eq. (2.7), with the right-hand side equal to zero, and yields to the standard method of analysis.

Let $\nu_0(t)$ be the solution of (2.29) for $k = 1$, and $\varphi_0(z)$, $\psi_0(z)$ potentials (2.18) corresponding to it. It is obvious that φ_0, ψ_0 give the solution of the first fundamental problem when $f(t) \equiv 0$, and hence, by virtue of the uniqueness of solution,

$$\varphi_0(z) = i\varepsilon z + \gamma, \qquad \psi_0(z) = -\bar{\gamma}; \qquad \gamma = \alpha + i\beta,$$

where ε, α, β are arbitrary real constants. Consequently, the following equalities are valid for any point z within S^+:

$$\frac{1}{2\pi i} \int_L \frac{\nu_0(t)\, dt}{t - z} = i\varepsilon z + \gamma, \qquad \frac{1}{2\pi i} \int_L \frac{\overline{\nu_0(t)} + \bar{t}\nu_0'(t)}{t - z}\, dt = \bar{\gamma}.$$

Hence, according to Cauchy's theorem,

$$\nu_0(t) - i\varepsilon t - \gamma = \varphi_*(t), \qquad \overline{\nu_0(t)} + \bar{t}\nu_0'(t) - \bar{\gamma} = \psi_*(t), \qquad (2.30)$$

where $\varphi_*(z)$, $\psi_*(z)$ are holomorphic functions in S^- vanishing at infinity: $\varphi_*(\infty) = \psi_*(\infty) = 0$. By eliminating $\nu_0(t)$ from (2.30), we can verify that the pair of functions, $\varphi_*(z)$, $-\psi_*(z)$ solve the first fundamental problem in S^- with the homogeneous conditions on L and, as just mentioned, become zero at infinity. From this it follows that $\varphi_*(z) = \psi_*(z) = 0$ everywhere in S^-, and from (2.30)

$$\nu_0(t) = i\varepsilon t + \alpha + i\beta. \tag{2.31}$$

Also, it is easy to see that in the case of the second fundamental problem $\varepsilon = 0$, and hence $\nu_0(t) = \alpha + i\beta$.

Thus, Eq. (2.29) and the homogeneous Muskhelishvili equation have the same set of non-trivial solutions.

According to the considerations developed above in the analysis of Eq. (2.21), the solutions of the form $\alpha + i\beta$ are no barrier to the solvability of (2.28) at either of the two possible values of k. Equation (2.28) in the case of the second fundamental problem is therefore always solvable. In the case of the first fundamental problem, however, the equation requires further investigation since the orthogonality condition corresponding to the solution it does not involve the couple of the external forces.

It is possible to try, following the conventional procedure, to eliminate the solution of the form it for the associated homogeneous equation by adding to the left-hand side of (2.28) the expression

$$-b\frac{d}{ds_0}\frac{1}{\bar{t}_0}, \qquad b = \frac{1}{2\pi}\int_L [\omega(t) - \overline{\omega(t)}]\,ds.$$

The modified equation may obviously be written as

$$-\frac{d}{ds}[\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)}] - b\frac{d}{ds}\frac{1}{\bar{t}} = Y_n - iX_n, \tag{2.32}$$

where $\varphi(t)$, $\psi(t)$ are boundary values of potentials (2.16) for $k = 1$. It is easy to prove (see, for example, N. I. Muskhelishvili [93], Sec. 102) that, if the condition for zero resultant couple of the external forces is fulfilled, it follows from (2.32) that $b = 0$, and hence, provided the above-mentioned condition is fulfilled, every solution of the new equation satisfies the original one.

By the argument used previously in the analysis of Eq. (2.29) it may be proved that the associated homogeneous equation modified in accordance with (2.32) has no solution different from $\alpha + i\beta$, provided only that the contour L satisfies the additional condition*

$$1 - \frac{1}{2\pi i}\int_L \frac{dt}{\bar{t}} \neq 0. \tag{2.33}$$

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* For all common types of regions encountered in applications the validity of the condition is established directly.

Thus, if the conditions of statics and conditions (2.33) are fulfilled, the modified equation (2.32) is solvable, the solution involves two arbitrary real constants and determines a regular solution of the first fundamental problem.

In conclusion, referring to the equations of the plane problem, the following may be mentioned. The Sherman-Lauricella equation is uniquely solvable and thereby compares favourably with other similar equations, including (2.21). On the other hand, if one proceeds from the equations of Par. 1°, then for the determination of a stress field regular in the closed region *it is necessary to find the first derivative of the solution, whereas in using Eqs.* (2.21) *and* (2.28) *it is sufficient, for the same purpose, to have the solution itself.* Besides (though it is not likely to be important), the order of smoothness of the contour L necessary for constructing a regular solution of the problem is lower by one in the present paragraph.

4°. The case of an infinite region. We proceed from the formulas

$$\left.\begin{aligned} \varphi(z;\ t) &= \frac{\omega(t)}{2\pi i}\ln(z-t)-\frac{\omega(t)}{2\pi i}\ln z,\\ \psi(z;\ t) &= \frac{\varkappa\overline{\omega(t)}}{2\pi i}\ln(z-t)-\frac{\omega(t)}{2\pi i}\,\frac{\bar{t}}{z-t}-\frac{\varkappa\overline{\omega(t)}}{2\pi i}\ln z, \end{aligned}\right\}\qquad(2.34)$$

where $\omega(t)$ is of the form of (2.13).

Formulas (2.34) describe a state of stress in an infinite plane obtained by the superposition of two elementary fields. One of these is produced by a concentrated force $(p,\ q)$ applied at the point $z=t$ of the unbounded plane, and the other by a force $(-p,\ -q)$ concentrated at the origin.

From (2.34), by integration over L, we determine two functions of z defining a regular field in S^- produced by a distributed load:

$$\varphi(z)=\frac{1}{2\pi i}\int\limits_L \omega(t)\ln\left(1-\frac{t}{z}\right)ds,$$

$$\psi(z)=\frac{\varkappa}{2\pi i}\int\limits_L \overline{\omega(t)}\ln\left(1-\frac{t}{z}\right)ds+\frac{1}{2\pi i}\int\limits_L\frac{\bar{t}\omega(t)\,ds}{t-z}\qquad(z\text{ in }S^-).$$

From this, as before, we construct two basic pairs of elastic potentials

$$\left.\begin{aligned} \varphi(z) &= \frac{1}{2\pi i}\int\limits_L \omega(t)\ln\left(1-\frac{t}{z}\right)ds,\\ \psi(z) &= \frac{k}{2\pi i}\int\limits_L \overline{\omega(t)}\ln\left(1-\frac{t}{z}\right)ds+\frac{1}{2\pi i}\int\limits_L\frac{\bar{t}\omega(t)\,ds}{t-z}\\ &\qquad (k=1,\ -\varkappa); \end{aligned}\right\}\qquad(2.35)$$

$$\left.\begin{aligned} \varphi(z) &= \frac{1}{2\pi i}\int\limits_L \omega(t)\ln\left(1-\frac{t}{z}\right)ds, \\ \psi(z) &= -\frac{k}{2\pi i}\int\limits_L \overline{\omega(t)}\ln\left(1-\frac{t}{z}\right)ds+\frac{1}{2\pi i}\int\limits_L \frac{\bar{t}\omega(t)\,ds}{t-z} \\ & \qquad\qquad\qquad\qquad (k=1,\ -\varkappa). \end{aligned}\right\} \qquad (2.36)$$

Potentials of higher orders are obtained from the above by differentiating with respect to the arc length s the kernels of the integral representations. *The functions* $\varphi(z)$, $\psi(z)$ *derivable in this way from* (2.35) *exactly reproduce the representation of D. I. Sherman.*

As in the preceding paragraph, it may be shown that functions (2.35) and (2.36) lead to the solution of the plane problem for S^- in the case when the stresses and rotation vanish at infinity and the resultant of the external forces applied to the boundary L is zero. The general case, as is known, can always be reduced to the preceding one.

The passage to the case of regions bounded by a finite number of closed curves presents no difficulty in principle.

3. General Mixed Problem*

We shall study the most general kind of mixed problem, relating to two-dimensional elasticity, when all the main types of boundary condition are realized on the same closed contour L which bounds a region of the medium. For definiteness, we shall consider the problem of the bending of elastic plates.** Assume that one part of the plate edge is clamped, another part is simply supported, and the remainder is free.

1°. **Preliminary remarks.** To begin with, let us explain some terms and concepts used in the theory of integral equations with Cauchy kernels (N. I. Muskhelishvili [92], N. P. Vekua [139]).

Let ab be a smooth unclosed arc with ends a and b, furnished with a definite positive sense which is taken from a to b. Let either of these end points be denoted by c.

Let $\varphi(t)$ be some function of the point t on ab. When saying that the function $\varphi(t)$ satisfies the Hölder condition on ab, or, in other words, belongs to the class H on ab, we mean that this condition is fulfilled on the closed arc ab; if $\varphi(t)$ satisfies the Hölder condition only in some neighbourhood of a given end c of the arc ab, including c, then $\varphi(t)$ is said to satisfy the Hölder condition near c.

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* The results of this section are presented in the author's paper [37].

** In the case of plane strain or generalized plane stress the general mixed problem should be represented as follows. The external forces are prescribed on a part of the boundary of the medium, and the remainder is in contact with a rigid body of given profile, the region of contact being divided into two portions: in the one portion there is adhesion (the frictional forces are large), and in the other there is no friction at all.

If the function $\varphi(t)$ satisfies the Hölder condition on every closed part of the arc ab, not containing ends, and if near either of the ends it is representable in the form

$$\varphi(t)=\frac{\varphi^*(t)}{(t-c)^{\alpha}}, \qquad 0\leqslant\alpha<1, \tag{3.1}$$

where $\varphi^*(t)$ satisfies the Hölder condition, then $\varphi(t)$ is said to belong to the class H^* on ab; if the preceding representation holds only near a given end c, then $\varphi(t)$ is said to belong to the class H^* near c. In the case when representation (3.1) holds for any arbitrarily small α, $\varphi(t)$ is said to belong to the class H_ε^* (on ab or near c respectively).

Let L be a smooth closed contour with a definite positive sense of description, and $a_1, \ldots, a_n$ some fixed points on it taken in the order in which we meet them when passing round L in the positive sense.

The function $\varphi(t)$ will be assumed given on L if it is given on each of the open arcs $a_k a_{k+1}$ $(k=1, \ldots, n;\ a_{n+1}=a_1)$. We shall say that the function $\varphi(t)$, given on L, belongs to the class H_a if there exist limits $\varphi(a_k-0)$, $\varphi(a_k+0)$ for all points a_k and if, on each of the closed arcs $a_k a_{k+1}$ taken separately, this function satisfies the Hölder condition provided it is assigned the values $\varphi(a_k+0)$, $\varphi(a_k-0)$ at the ends a_k, a_{k+1}. If the function $\varphi(t)$ satisfies the H^* (or H_ε^*) condition on each of the arcs $a_k a_{k+1}$, then $\varphi(t)$ is said to satisfy the H^* (or H_ε^*) condition on L.

Below we shall use N. I. Muskhelishvili's formulas describing the behaviour of the Cauchy-type integral near the ends of lines of integration (N. I. Muskhelishvili [92], Secs. 22 to 25). We present these formulas.

Let a function $\varphi(t)$, given on ab, satisfy the Hölder condition on every closed part of ab, not containing ends, and near the ends be representable in the form

$$\varphi(t)=\frac{\varphi^*(t)}{(t-c)^{\gamma}} \qquad (\gamma=\alpha+i\beta, \qquad 0\leqslant\alpha<1),$$

the expression $(t-c)^{\gamma}$ denoting a particular branch continually changing on ab.

Consider the Cauchy-type integral

$$\Phi(z)=\frac{1}{2\pi i}\int\limits_{ab}\frac{\varphi(t)\,dt}{t-z}.$$

Then the following estimates hold for points z close to c but not on ab:

I. If $\gamma = 0$, then

$$\Phi(z) = \pm \frac{\varphi(c)}{2\pi i} \ln \frac{1}{z-c} + \Phi_0(z), \tag{3.2}$$

where the upper sign is taken when $c = a$, and the lower sign when $c = b$. By $\ln \frac{1}{z-c}$ is meant any branch, single-valued near c in the plane cut along ab; $\Phi_0(z)$ denotes a bounded function having a definite limit as $z \to c$ along any path.

II. If $\gamma = \alpha + i\beta \neq 0$, then

$$\Phi(z) = \pm \frac{e^{\pm\gamma\pi i}}{2i \sin \gamma\pi} \frac{\varphi^*(c)}{(z-c)^\gamma} + \Phi_0(z), \tag{3.3}$$

where the signs are chosen as in Case I; $(z-c)^\gamma$ denotes a branch, single-valued near c in the plane cut along ab and assuming the value $(t-c)^\gamma$ on the left side of ab, and $\Phi_0(z)$ possesses the following properties: if $\alpha = 0$, it is bounded and tends to a definite limit as $z \to c$; if $\alpha > 0$, then

$$|\Phi_0(z)| < \frac{C}{|z-c|^{\alpha_0}},$$

where C and α_0 are real constants, with $\alpha_0 < \alpha$.

The following propositions hold for points $z = t$ situated on ab:

III. If $\gamma = 0$, then

$$\Phi(t) = \pm \frac{\varphi(c)}{2\pi i} \ln \frac{1}{t-c} + \Phi^*(t), \tag{3.4}$$

where $\Phi^*(t)$ satisfies the Hölder condition near c, including c; the signs are chosen as in Case I.

IV. If $\gamma = \alpha + i\beta \neq 0$, then

$$\Phi(t) = \frac{\pm \cot \gamma\pi}{2i} \frac{\varphi^*(c)}{(t-c)^\gamma} + \Phi^*(t), \tag{3.5}$$

with $\Phi^*(t)$ satisfying the Hölder condition near c if $\alpha = 0$; if, however, $\alpha > 0$, then

$$\Phi^*(t) = \frac{\Phi^{**}(t)}{|t-c|^{\alpha_0}},$$

where $\Phi^{**}(t)$ satisfies the Hölder condition near c (including c), and $\alpha_0 < \alpha$; the signs are chosen as above.

2°. **Transformation of boundary conditions and auxiliary formulas.** Suppose that the region S occupied by the middle surface of a thin isotropic elastic plate (a finite part of the plane of the variable $z = x + iy$) is bounded by a single closed contour L the Cartesian co-ordinates of whose points have continuous derivatives of the fourth order with respect to the arc length s. The positive sense on L is considered as one which leaves the region S on the left. The origin of co-ordinates is assumed lying inside the region S.

Let the boundary L be divided into N arcs by points $a_1, a_2, \ldots, a_n$ located successively when L is described in the positive sense. The set of unclosed arcs $\{a_k a_{k+1}\}$ is arbitrarily broken into three sets $L^{(1)}$, $L^{(2)}$, $L^{(3)}$ so that the arcs of each set will not have common ends. Let n_1, n_2, n_3 be the numbers of unclosed arcs $a_k a_{k+1}$ constituting, respectively, the sets $L^{(1)}$, $L^{(2)}$, $L^{(3)}$; it is clear that $n_1 + n_2 + n_3 = N$. Denote by $c_1, \ldots, c_{m_1}$ all common ends of the sets of arcs $L^{(1)}$ and $L^{(2)}$, by $b_1, \ldots, b_{m_2}$ the common ends of the sets $L^{(2)}$ and $L^{(3)}$, and by $d_1, \ldots, d_{m_3}$ the common ends of the sets $L^{(1)}$ and $L^{(3)}$. It is obvious that

$$m_1 = n_1 + n_2 - n_3, \qquad m_2 = n_2 + n_3 - n_1,$$
$$m_3 = n_3 + n_1 - n_2. \tag{3.6}$$

The end points b_k, c_k, d_k will be further called, respectively, the b-, c-, d-ends of the corresponding unclosed arcs.

Suppose that the part $L^{(1)}$ of the boundary L is fixed, $L^{(2)}$ is simply supported, and $L^{(3)}$ is free.

The problem of the equilibrium of such a plate leads, as noted in Sec. 1, to the determination of a regular biharmonic function $u(x, y)$ in the region S from the boundary conditions

$$\left.\begin{array}{lll} u = -w_0, & \dfrac{du}{dn} = -\dfrac{dw_0}{dn} & \text{on } L^{(1)}, \\ u = -w_0, & G(u) = -G(w_0) & \text{on } L^{(2)}, \\ G(u) = -G(w_0), & H(u) = -H(w_0) & \text{on } L^{(3)}. \end{array}\right\} \tag{3.7}$$

It is assumed that the above boundary conditions are fulfilled everywhere on the corresponding unclosed arcs, except possibly at the end points a_k. The desired solution $u(x, y)$ is required to have continuous partial derivatives of the first order in $S + L$. It is also assumed that the particular solution $w(x, y)$ of the bending equation (1.19) has continuous partial derivatives of the third order in the closed region.

The function $u(x, y)$ is represented, according to formula (1.21), in terms of the arbitrary holomorphic functions φ and χ, the former being subject to the condition

$$\operatorname{Im} \varphi'(0) = 0. \tag{3.8}$$

On the basis of formulas (1.7), (1.25), and (1.27) the boundary conditions of our problem (3.7) in terms of the potentials φ and ψ

then become

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = f(t) \quad \text{on } L^{(1)}, \tag{3.9}$$

$$\operatorname{Re}\left\{\frac{d}{dt}\left[-\varkappa\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)}\right]\right\} = \frac{1}{2(1-\nu)} G(w_0), \tag{3.10}$$

$$\operatorname{Re}\left\{\bar{t}'\left[\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)}\right]\right\} = -\frac{1}{2}\frac{dw_0}{ds} \quad \text{on } L^{(2)}, \tag{3.11}$$

$$\frac{d}{dt}\left[-\varkappa\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)}\right] =$$

$$= \frac{1}{2(1-\nu)}\left[G(w_0) + i\int^{s} H(w_0)\,ds\right] + iC \quad \text{on} \quad L^{(3)}. \tag{3.12}$$

Here C is a constant of integration,

$$f(t) = -\frac{1}{2}\left(\frac{\partial w_0}{\partial x} + i\frac{\partial w_0}{\partial y}\right) \quad \text{on} \quad L^{(1)},$$

$$\psi(z) = \chi'(z), \qquad t' = \frac{dt}{ds} = e^{i\vartheta}, \qquad \varkappa = \frac{3+\nu}{1-\nu}.$$

The integral on the right-hand side of equality (3.12) should be taken separately on each of the unclosed arcs of the set $L^{(3)}$ from some fixed point.

Thus, the problem of the elastic equilibrium of the bent plate has been reduced to the problem of the theory of analytic functions (3.9) to (3.12). In order to reduce the latter in turn to integral equations of the normal type, it is necessary first to integrate equalities (3.10) and (3.12) along each of the unclosed arcs of the sets $L^{(2)}$ and $L^{(3)}$ respectively. To avoid the appearance of excessive constants of integration, we shall proceed as follows.

Let $l_1, \ldots, l_n$ be all unclosed arcs which will remain after removal from the contour L of the set $L^{(1)}$ and of all those arcs of the set $L^{(3)}$ having d-ends on both sides; each arc l_j ($j = 1, \ldots, n$; $n \leqslant n_1$) represents, in general, a set of arcs from $L^{(2)}$ and $L^{(3)}$. All unclosed arcs from $L^{(2)}$ and $L^{(3)}$ contained in l_j are denoted, respectively, by $L_{r_j+1}, \ldots, L_{r_j+1_j}$ and $L_{m+p_j+1}, \ldots, L_{m+p_j+\pi_j}$, where

$$\left.\begin{aligned} &m = n_1 + n_2 - n, && r_1 = p_1 = 0, \\ &r_j = \rho_1 + \ldots + \rho_{j-1}, && p_j = \pi_1 + \ldots + \pi_{j-1}. \end{aligned}\right\} \tag{3.13}$$

It is obvious that ρ_k, π_k are some non-negative integers such that

$$\rho_1 + \ldots + \rho_n = n_2, \qquad \pi_1 + \ldots + \pi_n = n_3 - n_1 + n.$$

Let, further, $L_{n_2+1}, \ldots, L_{n_2+n_1-n}$ be all unclosed arcs from $L^{(3)}$ with both d-ends. Finally, we denote the initial points of the arcs l_j, L_k by a_{j0}, a_{k0} and the corresponding arc abscissas by s_{j0}, s_{k0} ($j = 1, \ldots, n$; $k = n_2 + 1, \ldots, n_2 + n_3$). Integrating expressions

(3.10) and (3.12) with respect to s and t along the corresponding unclosed arcs, we obtain

$$\operatorname{Re}\{\bar{t}'[-\varkappa\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}]+\int_{s_{j0}}^{s}\bar{t}''[\varkappa\varphi(t)-t\overline{\varphi'(t)}-\overline{\psi(t)}]\,ds=h_1(t)+\alpha_j \quad \text{on } L_k\ (k=r_j+1,\ldots,r_j+\rho_j;\ j=1,\ldots,n) \tag{3.14}$$

and

$$\left.\begin{aligned}&-\varkappa\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}=f(t)+iC_{k-n_2}t+\alpha_{k+n-n_2}+\\&\qquad\qquad+i\alpha_{k+n_1-n_2}\ \text{on}\ L_k\quad(k=n_2+1,\ldots,m),\\&-\varkappa\,\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}=f(t)+iC_{k-n_2}t+\alpha'_k+i\beta'_k\\&\qquad\qquad\text{on}\ L_k\quad(k=m+1,\ldots,n_2+n_3);\end{aligned}\right\}\tag{3.15}$$

here

$$\left.\begin{aligned}&h_1(t)=\frac{1}{2(1-\nu)}\int_{s_{j0}}^{s}G(w_0)\,ds\ \text{ on }\ l_j,\\&f(t)=\frac{1}{2(1-\nu)}\int_{a_{k0}}^{t}\left[G(w_0)+i\int_{s_{k0}}^{s}S(w_0)\,ds_1\right]dt\\&\qquad\text{on}\ L_k\qquad(k=n_2+1,\ldots,n_2+n_3)\end{aligned}\right\}\tag{3.16}$$

and C_s, α_j, α'_k, β'_k are undetermined real constants.

Note that all these constants will not generally be arbitrary. Indeed, it is easy to see that, except when $l_1, \ldots, l_n$ represent a set of all simply supported arcs (in this case $m = n_2 + n_3$), one of the constants α'_k, β'_k can always be expressed in terms of the others ($k = m + 1, \ldots, n_2 + n_3$). Namely,

$$\begin{aligned}\alpha'_k+i\beta'_k&=\lambda_k\alpha_{k+n_1-n_2}+\mu_kC_{k-n_2}+\nu_k\alpha_j+h_k+\\&+\nu_k\operatorname{Re}\int_{L_{k0}}\bar{t}''[-\varkappa\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}]\,ds\end{aligned}$$

$$(k=m+p_j+1,\ldots,m+p_j+\pi_j;\qquad j=1,\ldots,n).\tag{3.17}$$

Here $\alpha_{k+n_1-n_2}$, C_{k-n_2}, α_j are constants occurring on the right-hand sides of equalities (3.14), (3.15), and λ_k, μ_k, ν_k are some constants depending only on the positions of the points a_{k0} ; L_{k0} denotes some unclosed part of the contour L and

$$h_k=\frac{\nu_k}{2(1-\nu)}\int_{L_{k0}}G(w_0)\,ds\qquad(k=m+1,\ldots,n_2+n_3;\ j=1,\ldots,n).\tag{3.18}$$

Note, further, that all constants λ_k are different from zero, and ν_k is a real or a pure imaginary constant according as $\operatorname{Im}\lambda_k$ is different from zero or zero ($k = m + 1, \ldots, n_2 + n_3$).

From the preceding discussion it follows that the right-hand sides of equalities (3.14), (3.15) contain altogether $2n_3 + n_1$ arbitrary real constants which are to be determined in the process of solving the problem. These constants are α_j, C_k ($j = 1, \ldots, n_1 + n_3$; $k = 1, \ldots, n_3$).

Thus, the problem under consideration is reduced to the determination of two holomorphic functions in S, $\varphi(z)$ and $\psi(z)$, which satisfy conditions (3.9), (3.11), (3.14), (3.15) on the boundary of the region, by $\alpha_k' + i\beta_k'$ on the right-hand side of the second equality in (3.15) being meant expressions (3.17).

We now derive an integral formula which will be used below.

Note that the following equality is valid for any sufficiently regular biharmonic function $u(x, y)$ in $S + L$ (see, for example, Z. I. Khalilov [58]):

$$\int\limits_L \left[uH(u) - \frac{du}{dn}G(u)\right] ds =$$
$$= -\iint\limits_S \{\nu(\Delta u)^2 + (1-\nu)[u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2]\}\,dx\,dy, \qquad (3.19)$$

where H and G are the differential operators introduced above. If by $u(x, y)$ we understand here the right-hand side of formula (1.21), where $\varphi(z)$, $\chi(z)$ are arbitrary holomorphic functions in S which satisfy sufficient conditions for smoothness in $S + L$, and if we take into account that

$$\int\limits_L H(u)\,ds = 0,$$

then, as is easily seen, the left-hand side of equality (3.19) may be put in the form

$$\int\limits_L \left[uH(u) - \frac{du}{dn}G(u)\right] ds = 2(1-\nu)\operatorname{Im}\int\limits_L [\overline{\varphi(t)} + \bar{t}\varphi'(t) +$$
$$+ \psi(t)]\,d[-\varkappa\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)}]. \qquad (3.19\text{a})$$

Furthermore, from the same formula (1.21) we find after elementary computations

$$\frac{\nu}{1-\nu}(\Delta u)^2 + u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2 =$$
$$= 8\{2(\varkappa - 1)[\operatorname{Re}\varphi'(z)]^2 + |\bar{z}\varphi''(z) + \psi'(z)|^2\}.$$

On the basis of the three preceding equalities we derive the desired formula

$$\operatorname{Im}\int_L [\overline{\varphi(t)}+\bar{t}\varphi'(t)+\psi(t)]\,d\,[\varkappa\varphi(t)-t\overline{\varphi'(t)}-\overline{\psi(t)}]=$$

$$=4\iint_S \{2(\varkappa-1)[\operatorname{Re}\varphi'(z)]^2+|\bar{z}\varphi''(z)+\psi'(z)|^2\}\,dx\,dy \qquad (3.20)$$

which is valid for any pair of holomorphic functions $\varphi(z)$, $\psi(z)$ sufficiently regular in the closed region.

3°. **Reduction to integral equations.** The functions $\varphi(z)$ and $\psi(z)$ will be sought in the form of (2.10) when $k=-\varkappa$

$$\left.\begin{aligned}\varphi(z)&=\frac{1}{2\pi i}\int_L\frac{\omega(t)\,dt}{t-z},\\ \psi(z)&=-\frac{\varkappa}{2\pi i}\int_L\frac{\overline{\omega(t)}\,dt}{t-z}+\frac{1}{2\pi i}\int_L\frac{\omega(t)\,\overline{dt}}{t-z}-\frac{1}{2\pi i}\int_L\frac{\bar{t}\omega(t)\,dt}{(t-z)^2},\end{aligned}\right\}\qquad(3.21)$$

where $\omega(t)=\omega_1(t)+i\omega_2(t)$ is the unknown function of the point of the contour L which is Hölder continuous on L. The integrals are taken over both the complete contour and any part of it in the positive sense.

Substituting the boundary values of the functions $\varphi(z)$, $\varphi'(z)$, $\psi(z)$, defined by the preceding formulas, in conditions (3.9), (3.11), (3.14), (3.15), we have

$$\frac{1-\varkappa}{2}\omega(t_0)+\frac{1+\varkappa}{2\pi i}\int_L\frac{\omega(t)\,dt}{t-t_0}+K(t_0)=f(t_0)\ \text{ on }\ L^{(1)},\qquad(3.22)$$

$$\frac{1}{2}[\bar{t}_0'\omega(t_0)+t_0'\overline{\omega(t_0)}]-\frac{1}{\varkappa}\operatorname{Re}\left\{\bar{t}_0'K(t_0)+\int_{s_{j0}}^{s}\bar{t}''[\varkappa\omega(t)-K(t)]\,ds\right\}=$$

$$=-\frac{1}{\varkappa}[h_1(t_0)+\alpha_j],\qquad(3.23)$$

$$\frac{1-\varkappa}{2}[\bar{t}_0'\omega(t_0)+t_0'\overline{\omega(t_0)}]+\frac{1+\varkappa}{2\pi i}\int_L\frac{\bar{t}_0'\omega(t)-t_0'\overline{\omega(t)}}{t-t_0}\,dt+$$

$$+\frac{(1+\varkappa)\,t_0'}{2\pi i}\int_L\overline{\omega(t)}\,d\ln\frac{t-t_0}{\bar{t}-\bar{t}_0}+2\operatorname{Re}\{\bar{t}_0'K(t_0)\}=h_2(t_0)$$

$$\text{on } L_k\ (k=r_j+1,\ \ldots,\ r_j+\rho_j;\qquad j=1,\ \ldots,\ n),\qquad(3.24)$$

$$-\varkappa\omega(t_0)+K(t_0)=f(t_0)+iC_{k-n_2}t_0+\alpha_{k+n-n_2}+$$

$$+i\alpha_{k+n_1-n_2}\ \text{ on }\ L_k\ (k=n_2+1,\ \ldots,\ m),\qquad(3.25)$$

$$-\varkappa\omega(t_0)+K(t_0)+\nu_k \operatorname{Re}\int\limits_{L_{k0}} \bar{t}''[\varkappa\omega(t)-K(t)]\,ds=f(t_0)+$$
$$+h_k+(it_0+\mu_k)C_{k-n_2}+\lambda_k\alpha_{k+n_1-n_2}+\nu_k\alpha_j \text{ on } L_k \quad (k=m+$$
$$+p_j+1,\ \ldots,\ m+p_j+\pi_j;\qquad j=1,\ \ldots,\ n), \tag{3.26}$$

where

$$K(t_0)=\frac{\varkappa}{2\pi i}\int\limits_L \omega(t)\,d\ln\frac{\bar{t}-\bar{t}_0}{t-t_0}-\frac{1}{2\pi i}\int\limits_L \overline{\omega(t)}\,d\,\frac{t-t_0}{\bar{t}-\bar{t}_0},\quad h_2(t)=$$
$$=-\frac{1}{2}\frac{dw_0}{ds}. \tag{3.27}$$

The improper integrals occurring on the right-hand sides of equalities (3.22), (3.24) are to be understood as the Cauchy principal value.

The set of equalities (3.22) to (3.26) represents, in $\omega_1(t)$ and $\omega_2(t)$, a (real) system of singular integral equations with discontinuous coefficients. The right-hand side of this system contains, besides the known functions, the undetermined constants α_j and C_k.

On multiplying equality (3.23) by $\varkappa - 1$ and adding to equality (3.24), we obtain

$$\frac{1}{2\pi i}\int\limits_L \frac{\bar{t}_0'\omega(t)-t_0'\overline{\omega(t)}}{t-t_0}\,dt+\frac{\bar{t}_0'}{2\pi i}\int\limits_L \overline{\omega(t)}\,d\ln\frac{t-t_0}{\bar{t}-\bar{t}_0}+$$
$$+\operatorname{Re}\left\{\frac{1}{\varkappa}\bar{t}_0'K(t_0)+\frac{1-\varkappa}{(1+\varkappa)\varkappa}\int\limits_{s_{j0}}^{s_0}\bar{t}''[\varkappa\omega(t)-K(t)]\,ds\right\}=$$
$$=h^*(t_0)+\frac{1-\varkappa}{\varkappa(1+\varkappa)}\alpha_j, \tag{3.28}$$

where

$$h^*(t)=\frac{1}{1+\varkappa}\left[h_2(t)+\frac{1-\varkappa}{\varkappa}h_1(t)\right].$$

By separating the real and imaginary parts in equalities (3.22), (3.25), and (3.26) and using the matrix notation, we can write down this system of integral equations in the form of a single equality as follows:

$$A(t_0)\Omega(t_0)+\frac{B(t_0)}{\pi i}\int\limits_L \frac{\Omega(t)\,dt}{t-t_0}+M(t_0)=F(t_0)+\beta(t_0), \tag{3.29}$$

where $\Omega = (\omega_1, \omega_2)$ is the unknown vector (one-column matrix), A, B are known matrices, $\beta = (\beta_1, \beta_2)$ is a vector whose components contain the undetermined constants; written out in full these become

$$
\begin{aligned}
A &= \begin{Vmatrix} \frac{1}{2}(1-\varkappa) & 0 \\ 0 & \frac{1}{2}(1-\varkappa) \end{Vmatrix} & B &= \begin{Vmatrix} 0 & \frac{1}{2}i(1+\varkappa) \\ -\frac{1}{2}i(1+\varkappa) & 0 \end{Vmatrix} & &\text{on } L^{(1)}, \\
A &= \begin{Vmatrix} 0 & 0 \\ \cos\vartheta & \sin\vartheta \end{Vmatrix} & B &= \begin{Vmatrix} -i\sin\vartheta & i\cos\vartheta \\ 0 & 0 \end{Vmatrix} & &\text{on } L^{(2)}, \\
A &= \begin{Vmatrix} -\varkappa & 0 \\ 0 & -\varkappa \end{Vmatrix} & B &= \begin{Vmatrix} 0 & 0 \\ 0 & 0 \end{Vmatrix} & &\text{on } L^{(3)},
\end{aligned}
\tag{3.30}
$$

$$
\left.
\begin{aligned}
&\beta_1(t) + i\beta_2(t) = 0 \text{ on } L^{(1)}, \\
&\beta_1(t) + i\beta_2(t) = \frac{1}{\varkappa}\left[\frac{1-\varkappa}{1+\varkappa} - i\right]\alpha_j \\
&\qquad\qquad\qquad \text{on } L_k\,(k = r_j + 1, \ldots, r_j + \rho_j), \\
&\text{and} \\
&\beta_1(t) + i\beta_2(t) = iC_{k-n_2}t + \alpha_{k+n-n_2} + i\alpha_{k+n_1-n_2} \\
&\qquad\qquad\qquad \text{on } L_k\ (k = n_2 + 1, \ldots, m), \\
&\beta_1(t) + i\beta_2(t) = (it + \mu_k)\,C_{k-n_2} + \lambda_k\alpha_{k+n_1-n_2} + \nu_k\alpha_j \\
&\text{on } L_k\,(k = m + p_j + 1, \ldots, m + p_j + \pi_j;\quad j = 1, \ldots, n);
\end{aligned}
\right\}
\tag{3.31}
$$

$M = (M_1, M_2)$ denotes a vector whose components are Fredholm-type integral operators, and $F = (F_1, F_2)$ is the known vector whose components are expressed solely in terms of the particular solution ω_0 and vanish when $w_0 \equiv 0$. It is also easy to write down explicit expressions for the components of M, F. Denote

$$S^* = A + B. \qquad D^* = A - B, \qquad g = S^{*\,-1}D^*.$$

From expressions (3.30) we have

$$
\left.
\begin{aligned}
&g = \begin{Vmatrix} -\frac{1}{2\varkappa}(1+\varkappa^2) & \frac{i}{2\varkappa}(1-\varkappa^2) \\ \frac{i}{2\varkappa}(\varkappa^2-1) & -\frac{1}{2\varkappa}(\varkappa^2+1) \end{Vmatrix} \text{ on } L^{(1)}, \\
&g = \begin{Vmatrix} \cos 2\vartheta & \sin 2\vartheta \\ \sin 2\vartheta & -\cos 2\vartheta \end{Vmatrix} \text{ on } L^{(2)}, \qquad g = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} \text{ on } L^{(3)}.
\end{aligned}
\right\}
\tag{3.32}
$$

As can easily be verified, det $S^* \neq 0$, det $D^* \neq 0$ everywhere on L, and in consequence to system (3.29) may be applied the existing theory of systems of singular integral equations with discontinuous coefficients (N. P. Vekua [139]).

For the application of this theory it is necessary to determine the roots of the equation (E_0 is a unit matrix)

$$\det \{g^{-1}(t+0)\, g(t-0) - E_0\lambda\} = 0$$

at the points of discontinuity of the coefficients of the integral equations. These are in our case the points of change of the boundary conditions, a_k ($k = 1, \ldots, N$). From formula (3.32) we obtain after elementary computations

$$\left.\begin{aligned} &\det \{g^{-1}(b_k+0)\, g(b_k-0) - E_0\lambda\} = \lambda^2 - 1, \\ &\det \{g^{-1}(c_j+0)\, g(c_j-0) - E_0\lambda\} = \lambda^2 - 1, \\ &\det \{g^{-1}(d_s+0)\, g(d_s-0) - E_0\lambda\} = \lambda^2 + \frac{1+\varkappa^2}{\varkappa}\lambda + 1 \\ &(k = 1, \ldots, m_2; \qquad j = 1, \ldots, m_1; \qquad s = 1, \ldots, m_3). \end{aligned}\right\} \qquad (3.33)$$

Calculating now the index $\varkappa_0$ of the class H_ε^* of the system of integral equations (3.29) from a well-known formula (N. P. Vekua [139], formula (13.49), p. 116), we find, by virtue of formulas (3.33) and (3.6),

$$\varkappa_0 = -\left[m_3 + \frac{1}{2}(m_1 + m_2)\right] = -(n_1 + n_2). \qquad (3.34)$$

4°. **Investigation of integral equations.** It is known that the solutions of singular integral equations with discontinuous coefficients represent, in general, discontinuous functions. Bearing in mind the relation to the boundary value problem, we shall seek, according to the general theory of singular equations, the solutions of our system in the class H_ε^*, the best, from the viewpoint of smoothness, of all possible classes in which the solutions of such equations are to be sought. On the other hand, according to the requirements of the boundary value problem the desired solution $\omega(t)$ of Eq. (3.29) must be smoother than the functions of the class H_ε^* in the general case.

It will be shown later that any solution of the class H_ε^* of system (3.29) satisfies the Hölder condition on L, and its derivative belongs to the class H^* (on L).

The integral relations for $\omega(t)$ on the simply supported part of the contour will first be somewhat transformed.

Let $\Omega = (\omega_1, \omega_2)$ be a solution of the class H_ε^* of system (3.29) for some values of the constants α_j, C_k. By virtue of relation (3.28) we then have

$$\frac{1}{\pi i}\int\limits_{L_k} \frac{\bar{t}'\omega(t) - t'\overline{\omega(t)}}{t - t_0}\, dt = \Phi(t_0) \text{ on } L_k \quad (k = 1, 2, \ldots, n_2), \qquad (3.35)$$

where $\Phi_0(t)$ stands for the following expression:

$$\Phi_0(t_0) = -\sum_{p=1}^{N}{}'_{k} \frac{1}{\pi i} \int_{L_p} \frac{\bar{t}'\omega(t) - t'\overline{\omega(t)}}{t - t_0}\, dt + \Phi_0(t_0), \tag{3.36}$$

$$\Phi_0(t_0) = \frac{1}{\pi i} \int_L \frac{(\bar{t}' - \bar{t}'_0)\,\omega(t) - (t' - t'_0)\,\overline{\omega(t)}}{t - t_0}\, dt -$$

$$-\frac{\bar{t}'_0}{\pi i} \int_L \overline{\omega(t)}\, d \ln \frac{t - t_0}{\bar{t} - \bar{t}_0} - 2\,\mathrm{Re}\left\{\frac{1}{\varkappa} \bar{t}'_0 K(t_0) + \frac{1 - \varkappa}{\varkappa(1 + \varkappa)} \int_{s_{j0}}^{s} \bar{t}''\,[\varkappa\omega(t) - \right.$$

$$\left. - K(t)]\, ds\right\} + 2h^*(t_0) + \frac{2(1 - \varkappa)}{\varkappa(1 + \varkappa)}\,\alpha_j \text{ on } L_k \ (k = r_j +$$

$$+ 1, \ldots, r_j + \rho_j; \qquad j = 1, \ldots, n) \tag{3.37}$$

(the signs to the right of the symbol Σ denote that the corresponding value of the summation index is left out).

Inverting Eq. (3.35) and taking into account that $\bar{t}'\omega(t) - t'\overline{\omega(t)}$ is an almost bounded function on L_k, we obtain

$$\bar{t}'_0\omega(t_0) - t'_0\overline{\omega(t_0)} = \frac{R_k(t_0)}{\pi i} \int_{L_k} \frac{\Phi(t)\, dt}{R_k(t)\,(t - t_0)} \text{ on } L_k \ (k = 1, \ldots, n_2), \tag{3.38}$$

where

$$R_k(t) = \sqrt{(t - p_k)(t - q_k)},$$

with p_k, q_k denoting the end points of the arc L_k; by $R_k(t)$ is meant the boundary value assumed by the radical $\sqrt{(z - p_k)(z - q_k)}$ on L_k from the left.

If (3.36) is substituted in formula (3.38), among other expressions there will be an expression of the form

$$I_{k,p}(t_0) = -\frac{R_k(t_0)}{\pi i} \int_{L_k} \frac{dt_1}{R_k(t_1)(t_1 - t_0)} \left\{\frac{1}{2\pi i} \int_{L_p} \frac{\bar{t}'\omega(t) - t'\overline{\omega(t)}}{t - t_1}\, dt\right\}$$

$$t_0 \in L_k \ (k \neq p).$$

We have

$$I_{k,p}(t_0) = \frac{1}{2\pi i} \int_{L_p} \frac{\bar{t}'\omega(t) - t'\overline{\omega(t)}}{t - t_0}\, dt - \frac{R_k(t_0)}{\pi i} \times$$

$$\times \lim_{z \to t_0} \int_{L_k} \frac{1}{R_k(t_1)} \left\{\frac{1}{2\pi i} \int_{L_p} \frac{\bar{t}'\omega(t) - t'\omega(t)}{t - t_1}\right\} \frac{dt_1}{t_1 - z}, \tag{3.39}$$

with z denoting a point not on L_k and approaching the point t_0 from the left. Introducing the function

$$P_p(z)=\frac{1}{2\pi i}\int\limits_{L_p}\frac{\bar{t}'\omega(t)-t'\overline{\omega(t)}}{t-z}\,dt$$

and noting that the ratio $\frac{P_p(z)}{R_k(z)}$ is a holomorphic function in any region not containing points of the arcs L_k, L_p, with the use of Cauchy's formula we obtain

$$\frac{1}{2\pi i R_k(z)}\int\limits_{L_p}\frac{\bar{t}'\omega(t)-t'\overline{\omega(t)}}{t-z}\,dt=\frac{1}{2\pi i}\int\limits_{\Lambda_{k,\,p}}\frac{P_p(t)dt}{R_k(t)(t-z)}\,; \tag{3.40}$$

here $\Lambda_{k,p}$ denotes the set of two closed simple contours Λ_k, Λ_p enclosing, respectively, the arcs L_k, L_p, passing round them in the clockwise direction and so close to them that the point z will be outside either of the contours Λ_k, Λ_p. Taking into account the multiple-valuedness of the radical $R_k(t)$ and also the equality

$$P_p^+(t)-P_p^-(t)=\bar{t}'\omega(t)-t'\overline{\omega(t)}\ \text{ on } L_p,$$

we find

$$\frac{1}{2\pi i}\int\limits_{\Lambda_{k,\,p}}\frac{P_p(t)\,dt}{R_k(t)(t-z)}=\frac{1}{2\pi i}\int\limits_{L_p}\frac{\bar{t}'\omega(t)-t'\overline{\omega(t)}}{R_k(t)(t-z)}\,dt+$$

$$+\frac{1}{\pi i}\int\limits_{L_k}\frac{1}{R_k(t_1)}\left\{\frac{1}{2\pi i}\int\limits_{L_p}\frac{\bar{t}'\omega(t)-t'\omega(t)}{t-t_1}\,dt\right\}\frac{dt_1}{t_1-z}\,.$$

The latter equality, together with (3.40), gives

$$\frac{1}{\pi i}\int\limits_{L_k}\frac{1}{R_k(t_1)}\left\{\frac{1}{2\pi i}\int\limits_{L_p}\frac{\bar{t}'\omega(t)-t'\overline{\omega(t)}}{t-t_1}\right\}\frac{dt_1}{t_1-z}=$$

$$=\frac{1}{2\pi i R_k(z)}\int\limits_{L_p}\frac{\bar{t}'\omega(t)-t'\overline{\omega(t)}}{t-z}\,dt-\frac{1}{2\pi i}\int\limits_{L_p}\frac{\bar{t}'\omega(t)-t'\overline{\omega(t)}}{R_k(t)(t-z)}\,dt.$$

Substituting this expression in (3.39) and passing to the limit as $z\to t_0$ on L_k, we obtain

$$I_{k,\,p}(t_0)=\frac{R_k(t_0)}{2\pi i}\int\limits_{L_p}\frac{\bar{t}'\omega(t)-t'\overline{\omega(t)}}{R_k(t)(t-z)}\,dt. \tag{3.41}$$

Substitution of expression (3.36) in formula (3.38), with the preceding equality taken into account, gives

$$\bar{t}_0'\omega(t_0)-t_0'\overline{\omega(t_0)}=\sum_{p=1}^{N}{}'_{k}\,\frac{R_k(t_0)}{\pi i}\int\limits_{L_p}\frac{\bar{t}'\omega(t)-t'\overline{\omega(t)}}{R_k(t)(t-t_0)}\,dt+$$

$$+\frac{R_k(t_0)}{\pi i}\int\limits_{L_k}\frac{\Phi_0(t)\,dt}{R_k(t)(t-t_0)}\ \text{ on } L_k\ (k=1,\ \ldots,\ n_2). \tag{3.42}$$

Let L_j be an unclosed arc adjacent to L_k. We have

$$\sum_{p=1}^{N}{}'_k \frac{1}{\pi i}\int_{L_p} \frac{\overline{t'}\omega(t)-t'\overline{\omega(t)}}{R_k(t)(t-t_0)}\,dt=\frac{1}{\pi i}\int_{L_j}\frac{\overline{t'}\omega(t)-t'\overline{\omega(t)}}{R_k(t)(t-t_0)}\,dt+$$
$$+\sum_{p=1}^{N}{}'_{k,\,j}\frac{1}{\pi i}\int_{L_p}\frac{\overline{t'}\omega(t)-t'\overline{\omega(t)}}{R_k(t)(t-t_0)}\,dt. \qquad (3.43)$$

Consider the integral

$$I_j(t_0)=\frac{1}{\pi i}\int_{L_j}\frac{\tau(t)\,dt}{R_k(t)(t-t_0)} \qquad (t_0 \text{ on } L_k),$$

where $\tau(t)$ is some function of the point t on L which belongs to the class H^*. Clearly, we have

$$I_j(t_0)=\frac{1}{\pi i}\int_{L_j}\frac{\tau(t)-\tau(t_0)}{R_k(t)(t-t_0)}\,dt+\frac{\tau(t_0)}{\pi i}\int_{L_j}\frac{dt}{R_k(t)(t-t_0)}. \qquad (3.44)$$

Representing the last integral as

$$\frac{1}{\pi i}\int_{L_j}\frac{dt}{R_k(t)(t-t_0)}=\sum_{p=1}^{N}{}'_k\frac{1}{\pi i}\int_{L_p}\frac{dt}{R_k(t)(t-t_0)}-$$
$$-\sum_{p=1}^{N}{}'_{k,\,j}\frac{1}{\pi i}\int_{L_p}\frac{dt}{R_k(t)(t-t_0)} \qquad (3.45a)$$

and noting that

$$\sum_{p=1}^{N}{}'_k\frac{1}{\pi i}\int_{L_p}\frac{dt}{R_k(t)(t-t_0)}=\frac{1}{R_k(t_0)} \quad (\text{on } L_k), \qquad (3.45)$$

expression (3.44) may be written in the form

$$I_j(t_0)=\frac{1}{\pi i}\int_{L_j}\frac{\tau(t)-\tau(t_0)}{R_k(t)(t-t_0)}\,dt+$$
$$+\tau(t_0)\left[\frac{1}{R_k(t_0)}-\sum_{p=1}^{N}{}'_{k,\,j}\frac{1}{\pi i}\int_{L_p}\frac{dt}{R_k(t)(t-t_0)}\right]. \qquad (3.46)$$

Let, now, L_k be an arc from $L^{(2)}$ having at least one of the end points (say q_k) as a c-end, and L_j an arc from $L^{(1)}$ with one of the ends q_k. By virtue of equality (3.22) on L_j we have

$$\bar{t}_0'\omega(t_0)-t_0'\overline{\omega(t_0)}=-\frac{1+\varkappa}{(1-\varkappa)\pi i}\int\limits_L\frac{\bar{t}'\omega(t)-t'\overline{\omega(t)}}{t-t_0}\,dt+$$

$$+\frac{1+\varkappa}{(1-\varkappa)\pi i}\int\limits_L\frac{(\bar{t}'-\bar{t}_0')\,\omega(t)+(t'-t_0')\,\overline{\omega(t)}}{t-t_0}\,dt+$$

$$+\frac{(1+\varkappa)\,t_0'}{(1-\varkappa)\pi i}\int\limits_L\omega(t)\,d\ln\frac{t-t_0}{\bar{t}-\bar{t}_0}+\frac{4i}{1-\varkappa}\operatorname{Im}\{\bar{t}_0'\,[f(t_0)-K(t_0)]\}. \qquad (3.47)$$

We now introduce, following D. I. Sherman [120], a function $p(t)$ defined on the entire boundary L:

$$p(t)=\begin{cases} f(t) & \text{when } t \text{ is on } L^{(1)}, \\ P(t) & \text{when } t \text{ is on } L-L^{(1)}; \end{cases} \qquad (3.48)$$

$P(t)$ denotes a polynomial in t chosen so that the function $p(t)$ will be continuous on L together with its derivatives up to the second order.

By virtue of formulas (3.46) to (3.48) the first integral on the right-hand side of expression (3.43) then becomes

$$\frac{1}{\pi i}\int\limits_{L_j}\frac{\bar{t}'\omega(t)-t'\overline{\omega(t)}}{R_k(t)(t-t_0)}\,dt=-\left\{\frac{1+\varkappa}{(1-\varkappa)\pi i}\int\limits_L\frac{\bar{t}'\omega(t)+t'\overline{\omega(t)}}{t-t_0}\,dt+\right.$$

$$\left.+K_0(t_0)\right\}\left\{\frac{1}{R_k(t_0)}-\sum_{p=1}^{N}{}'_{k,\,j}\,\frac{1}{\pi i}\int\limits_{L_p}\frac{dt}{R_k(t)(t-t_0)}\right\}-$$

$$-\frac{1}{\pi i}\int\limits_{L_j}\frac{K_0(t)-K_0(t_0)}{R_k(t)(t-t_0)}\,dt+\frac{1+\varkappa}{(1-\varkappa)\pi i}\int\limits_L\frac{\bar{t}'\omega(t)+t'\overline{\omega(t)}}{t-t_0}\times$$

$$\times\left\{\frac{1}{\pi i}\int\limits_{L_j}\frac{dt_1}{R_k(t_1)(t_1-t)}\right\}dt \text{ on } L_k, \qquad (3.49)$$

where

$$K_0(t_0)=\frac{1+\varkappa}{(\varkappa-1)\pi i}\int\limits_L\frac{(\bar{t}'+\bar{t}_0')\,\omega(t)+(t'-t_0')\,\overline{\omega(t)}}{t-t_0}\,dt+$$

$$+\frac{(1+\varkappa)\,t_0'}{(\varkappa-1)\pi i}\int\limits_L\overline{\omega(t)}\,d\ln\frac{t-t_0}{\bar{t}-\bar{t}_0}+\frac{4i}{\varkappa-1}\operatorname{Im}\{\bar{t}_0'\,[p(t_0)-K(t_0)]\}.$$

Inserting this expression with (3.43) in formula (3.42), we obtain

$$
\begin{aligned}
&\overline{t_0'}\omega(t_0)-t_0'\overline{\omega(t_0)}+\frac{1+\varkappa}{(1-\varkappa)\pi i}\int_L \frac{\overline{t'}\omega(t)+t'\overline{\omega(t)}}{t-t_0}\,dt+K(t_0)=\\
&\quad=R_k(t_0)\left\{\frac{1+\varkappa}{(1-\varkappa)\pi i}\int_L \frac{\overline{t'}\omega(t)+t'\overline{\omega(t)}}{t-t_0}\left[\frac{1}{\pi i}\int_{L_j}\frac{dt_1}{R_k(t_1)(t_1-t)}\right]dt+\right.\\
&\quad+\frac{1}{\pi i}\int_{L_k}\frac{\Phi_0(t)\,dt}{R_k(t)(t-t_0)}+\sum_{p=1}^{N}{}'_{k,j}\,\frac{1}{\pi i}\int_{L_p}\frac{\overline{t'}\omega(t)-t'\overline{\omega(t)}}{R_k(t)(t-t_0)}\,dt-\\
&\quad\left.-\frac{1}{\pi i}\int_{L_j}\frac{K_0(t)-K_0(t_0)}{R_k(t)(t-t_0)}\,dt\right\}+\left\{\frac{1+\varkappa}{(1-\varkappa)\pi i}\int_L\frac{\overline{t'}\omega(t)+t'\overline{\omega(t)}}{t-t_0}\,dt+\right.\\
&\qquad\left.+K_0(t_0)\right\}\sum_{p=1}^{N}{}'_{k,j}\,\frac{R_k(t_0)}{\pi i}\int_{L_p}\frac{dt}{R_k(t)(t-t_0)}\quad\text{on } L_k.
\end{aligned}
\tag{3.50}
$$

It is easy to prove that, under the assumptions made above regarding the contour L and the given functions on it, the right-hand side of the foregoing equality vanishes at the point q_k; in the neighbourhood of this point, however, it satisfies the Hölder condition, and its first derivative belongs to the class H^*.

In view of the fact that the function $K_0(t)$ is twice continuously differentiable on L, it is sufficient to prove our statement for the first two terms on the right-hand side of equality (3.50).

For this purpose it will be found helpful to refer to the following property of the Cauchy-type integral of the special kind.

Let $a_1, a_2, \ldots, a_n$ be arbitrary points on the curve L, and $F(t)$ a given function on it which has discontinuities of the first kind at the points a_k and such that $F(t)\in H_a$, $F'(t)\in H^*$ on L. Assume

$$R(z)=\prod_{k=1}^{n}(z-a_k)^{\frac{1}{2}+i\varepsilon_k}$$

(ε_k are some real numbers) and consider the Cauchy-type integral

$$F(z)=\frac{R(z)}{2\pi i}\int_L\frac{F(t)\,dt}{R(t)(t-z)},\tag{3.51}$$

where z is any point not lying on L.

Then the limiting values of the function $F(z)$ on L from the inside and outside satisfy the Hölder condition, $F^+(t)$, $F^-(t)\in H$, and the limiting values of the derivative $F'(z)$ belong to the class H^*:

$$\left(\frac{dF}{dz}\right)^+=\frac{dF^+}{dt}\in H^*,\qquad\left(\frac{dF}{dz}\right)^-=\frac{dF^-}{dt}\in H^*.$$

The assertion regarding the functions $F^+(t)$, $F^-(t)$ is an almost direct consequence of formulas (3.3) and (3.5). As to the statement regarding the limiting values of the derivative, its proof requires a more elaborate reasoning (G. F. Mandzhavidze [77], pp. 289-290).

We return to Eq. (3.50).

As is seen from expression (3.37), $\Phi_0'(t) \in H^*$ (more precisely, $\Phi_0'(t) \in H_\varepsilon^*$) on L_k ($k = 1, \ldots, n_2$). Hence, our statement regarding the second term on the right-hand side of Eq. (3.50) follows immediately from formulas (3.5) and the foregoing proposition. It remains to examine the integral

$$J(t_0) = \frac{R_k(t_0)}{\pi i} \int_L \frac{\overline{t'\omega(t)} + t'\overline{\omega(t)}}{t - t_0} \left\{ \frac{1}{\pi i} \int_{L_j} \frac{dt_1}{R_k(t_1)(t_1 - t)} \right\} dt. \tag{3.52}$$

By virtue of formulas (3.3) and (3.5) we have near q_k

$$\left.\begin{aligned} \frac{1}{\pi i} \int_{L_j} \frac{dt_1}{R_k(t_1)(t_1 - t)} &= \frac{c_0}{\sqrt{t - q_k}} + \frac{\varphi^*(t)}{|t - q_k|^{\alpha_0}} \quad \text{on } L_k, \\ \frac{1}{\pi i} \int_{L_j} \frac{dt_1}{R_k(t_1)(t_1 - t)} &= \frac{\varphi^{**}(t)}{|t - q_k|^{\alpha_0}} \quad \text{on } L_j, \end{aligned}\right\} \tag{3.53}$$

where c_0 is a constant, $\varphi^*(t)$, $\varphi^{**}(t)$ satisfy the Hölder condition near q_k, and α_0 is a positive real constant less than 1/2. The expression $\overline{t'\omega(t)} + t'\overline{\omega(t)}$, being a function of the class H_ε^* on L, satisfies the Hölder condition on L_k by virtue of relation (3.23). Hence, on the basis of the same formulas (3.3) and (3.5) it is easily concluded that $J(t)$ vanishes at the point q_k and satisfies the Hölder condition near it.

Assume further $J(t) = J_1(t) + J_2(t)$, where

$$J_1(t_0) = \frac{R_k(t_0)}{\pi i} \int_{L_k} \frac{\overline{t'\omega(t)} + t'\overline{\omega(t)}}{t - t_0} \left\{ \frac{1}{\pi i} \int_{L_j} \frac{dt_1}{R_k(t_1)(t_1 - t)} \right\} dt,$$

$$J_2(t_0) = \frac{R_k(t_0)}{\pi i} \int_{L - L_k} \frac{\overline{t'\omega(t)} + t'\overline{\omega(t)}}{t - t_0} \left\{ \frac{1}{\pi i} \int_{L_j} \frac{dt_1}{R_k(t_1)(t_1 - t)} \right\} dt.$$

By using formulas (3.45a) and (3.45), we rewrite the first integral as

$$J_1(t_0) = \frac{R_k(t_0)}{\pi i} \int_{L_k} \frac{\overline{t'\omega(t)} + t'\overline{\omega(t)}}{R_k(t)(t - t_0)} dt -$$

$$- \frac{R_k(t_0)}{\pi i} \int_{L_k} \frac{\overline{t'\omega(t)} + t'\overline{\omega(t)}}{t - t_0} \left\{ \sum_{p=1}^{N}{}'_{k,j} \frac{1}{\pi i} \int_{L_p} \frac{dt_1}{R_k(t_1)(t_1 - t)} \right\} dt \tag{3.54}$$

and note that, by equality (3.23), the function $\frac{d}{dt}[\overline{t'}\omega(t)+t'\overline{\omega(t)}]$ belongs to the class H_ε^* on L_k. Then, on the basis of the foregoing results in relation to an integral of the form of (3.51), the derivative of the first term on the right-hand side of expression (3.54) belongs to the class H^* on L_k. The truth of the statement regarding the second term in expression (3.54) is self-evident.

It now remains to show that the expression

$$J_*(t_0)=\frac{R_k(t_0)}{\pi i}\int\limits_{L-L_k}\frac{\overline{t'}\omega(t)+t'\overline{\omega(t)}}{(t-t_0)^2}\left\{\frac{1}{\pi i}\int\limits_{L_j}\frac{dt_1}{R_k(t_1)(t_1-t)}\right\}dt$$

belongs to the class H^* on L_k because a similar statement for the second term of the derivative of $J_2(t)$ follows directly from formula (3.3) if the second equality of (3.53) is taken into account.

Considering equality (3.53) and following N. I. Muskhelishvili ([92], Sec. 23), it can easily be shown that near q_k

$$|J_*(t)|<\frac{\text{constant}}{|t-q_k|^\beta},\qquad |J_2'(t)|<\frac{\text{constant}}{|t-q_k|^{\beta+1}}\qquad (t\in L_k),$$

(where β is any real number such that $\beta>\alpha_0+1/2$. It is also easy to verify (N. I. Muskhelishvili [92], Sec. 6) that the function $|t-q_k|^{\beta+\varepsilon}J_*(t)$ satisfies the Hölder condition near q_k for any arbitrarily small $\varepsilon>0$. On the other hand, in view of the fact that $\alpha_0<1/2$ the numbers β and ε can be chosen so as to satisfy the inequality $\beta+\varepsilon<1$, and hence $J_*(t)\in H^*$ on L_k. Our statement regarding the right-hand side of expression (3.50) is completely proved.*

Multiplying equality (3.50) by $\frac{1}{2}(1-\varkappa)$, adding to equality 3.24) and taking into account relation (3.49), we obtain

$$\frac{1-\varkappa}{2}\omega(t_0)+\frac{1+\varkappa}{2\pi i}\int\limits_L\frac{\omega(t)\,dt}{t-t_0}+K(t_0)=M_1(t_0)\quad\text{on } L_k,\qquad(3.55)$$

where $M_1(t)$ satisfies the Hölder condition near q_k and the derivative $M_1'(t)$ belongs to the class H^*. Furthermore, as is easily seen from the foregoing, the sum of the first two terms in equality (3.55) remains continuous in passing through the point q_k, i.e., $\lim\limits_{L_k\ni t\to q_k}M_1(t)=$
$=\lim\limits_{L_j\ni t\to q_k}f(t)$.

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* In the case when both points, p_k, q_k, are c-ends of the arc L_k we obtain the same expression for the left-hand side of equality (3.50), the only difference being that k, j, s, rather than k, j, will be excluded from the summation, and the integration will be carried out over the arc L_j+L_s instead of over L_j; L_s is an unclosed arc from $L^{(1)}$ having for either of its ends the point p_k. It is obvious that in this case the statement regarding the right-hand side of equality (3.50) will be true for both ends of the arc L_k.

Consequently, if we restrict our attention to the solution of the class H_ε^* of system (3.29), the integral relations on the simply supported part (3.23) and (3.24), in the presence of a c-end in the arc L_k, may be reduced to the form (3.55).

In the case when q_k (or p_k) is a b-end of the arc L_k from $L^{(2)}$ the equalities (3.23) and (3.24) are much easier to transform to the required form. This time L_j, an adjacent arc to L_k having for one of its ends q_k, will be a free part of the contour L, and hence we shall have on it the Fredholm-type equation (3.25) or (3.26) instead of equality (3.22). In this case, by an argument quite similar to that used previously, we can reduce relations (3.23) and (3.24) to a single equality of the form

$$-\varkappa\omega(t_0)+K(t_0)=M_2(t_0) \text{ on } L_k \qquad (k=1,\ldots,n_2), \tag{3.56}$$

where $M_2(t)$ satisfies the Hölder condition near q_k and $M_2'(t)$ belongs in the same neighbourhood to the class H^*. Moreover, the solution $\omega(t)$ will be continuous in passing through the point q_k. If, however, both ends of L_k are b-ends, the above-mentioned properties of the functions $M_2(t)$, $M_2'(t)$ remain entirely the same on L_k, and $\omega(t)$ will be continuous at both ends (for additional details see the author's paper [37]).

We further introduce new points of discontinuity on L as follows.

Let $L_1^{(1)}$ and $L_1^{(3)}$ be sets of all arcs from $L^{(2)}$ with the two c- and b-ends, respectively, $l_1^{(0)},\ldots,l_{m_0-m_3}^{(0)}$ the remaining arcs of $L^{(2)}$ (m_0 is an integer, $m_0\geqslant m_3$). Divide each arc $l_j^{(0)}$ ($j=1,\ldots,m_0-m_3$) by its interior point d_{m_3+j} into two parts, and denote the sets of these parts with ends, respectively, c_k and b_k by $L_2^{(1)}$ and $L_2^{(3)}$. Assume

$$L_0^{(1)}=L^{(1)}+L_1^{(1)}+L_2^{(1)}, \qquad L_0^{(3)}=L^{(3)}+L_1^{(3)}+L_2^{(3)}.$$

Then, by virtue of the foregoing, the solution of the class H_ε^* of system (3.29) will satisfy an equation of the form

$$a(t_0)\omega(t_0)+\frac{b(t_0)}{\pi i}\int\limits_L\frac{\omega(t)\,dt}{t-t_0}=M_0(t_0), \tag{3.57}$$

where

$$a(t)=\begin{cases}\frac{1}{2}(1-\varkappa) & \text{on } L_0^{(1)}\\ -\varkappa & \text{on } L_0^{(3)}\end{cases} \qquad b(t)=\begin{cases}\frac{1}{2}(1+\varkappa) & \text{on } L_0^{(1)},\\ 0 & \text{on } L_0^{(3)}.\end{cases}$$

$M_0(t)$ is some function on L such that $M_0(t)\in H_d$, $M_0'(t)\in H^*$; $d_1,\ldots,d_{m_0}$ and $a_1,\ldots,a_N$, $d_{m_3+1},\ldots,d_{m_0}$ exhaust all points of discontinuity of the functions $M_0(t)$ and $M_0'(t)$, respectively; equality (3.57) holds everywhere on L except possibly at the points $d_1,\ldots,d_{m_3}$.

By relation (3.57), the sectionally holomorphic function $\varphi(z)$ defined in terms of $\omega(t)$ by the formula

$$\varphi(z)=\frac{1}{2\pi i}\int\limits_L \frac{\omega(t)\,dt}{t-z}$$

will be represented in the form (N. I. Muskhelishvili [91])

$$\varphi(z)=\frac{R(z)}{2\pi i}\int\limits_L \frac{M^*(t)\,dt}{R(t)(t-z)}, \tag{3.58}$$

where

$$M^*(t)=\frac{M_0(t)}{a(t)+b(t)}, \qquad R(z)=\prod_{k=1}^{m_0}(z-d_k)^{\frac{1}{2}+i\varepsilon_k},$$

$$\varepsilon_k=\pm\frac{1}{2\pi}\ln\varkappa.$$

This integral has the form of the Cauchy-type integral (3.51), and hence, by virtue of the foregoing, we conclude: $\varphi^+(t)$, $\varphi^-(t)\in H$, $\frac{d\varphi^+}{dt}$, $\frac{d\varphi^-}{dt}\in H^*$ on L. Consequently, $\omega(t)\in H$, $\omega'(t)\in H^*$ on L, which was to be proved. Moreover, near the points of discontinuity we have (a denotes any one of the points $a_1, \ldots, a_N, d_{m_1+1}, \ldots, d_{m_0}$)

$$|\varphi'(z)|<\frac{\text{constant}}{|z-a|^{\alpha}}, \qquad 0\leqslant\alpha<1. \tag{3.59}$$

By virtue of the just proved property of the solution of the class H_ε^* to system (3.29), from formulas (3.21) we have for such solutions

$$z\overline{\varphi'(z)}+\overline{\psi(z)}=\varkappa\varphi(z)+\frac{\varkappa}{2\pi i}\int\limits_L \omega(t)\,d\ln\frac{\bar t-\bar z}{t-z}+\frac{1}{2\pi i}\int\limits_L \overline{\omega(t)}\,d\frac{t-z}{\bar t-\bar z}.$$

From this it follows immediately that the functions $z\overline{\varphi'(z)}+\overline{\psi(z)}$ and $z\overline{\varphi''(z)}+\overline{\psi'(z)}$ possess in $S+L$ the same properties as the functions $\varphi(z)$ and $\varphi'(z)$, respectively. In particular, the limiting values on L of the function $z\overline{\varphi''(z)}+\overline{\psi'(z)}$ belong to the class H^*, and an estimate of the form of (3.59) is valid for it near the points of discontinuity.

From the preceding discussion it can also be seen that if $\omega(t)$ is a solution of the class H_ε^* to system (3.29), then formula (3.20) is valid for functions $\varphi(z)$, $\psi(z)$ corresponding to it by formulas (3.21).

5°. **Existence of solution.** We proceed to the question of the existence of a solution to the problem. Consider the homogeneous equations

$$A(t_0)\Omega(t_0)+\frac{B(t_0)}{\pi i}\int\limits_L \frac{\Omega(t)\,dt}{t-t_0}+M(t_0)=0. \tag{3.60}$$

Let $\Omega_0 = (\omega_1^{(0)}, \omega_2^{(0)})$ be a solution of the class H_ε^* to system (3.60), and $\varphi_0(z)$, $\psi_0(z)$ the corresponding holomorphic functions. For these functions we have the following contour conditions:

$$\varphi_0(t) + t\overline{\varphi_0'(t)} + \overline{\psi_0(t)} = 0 \text{ on } L^{(1)}, \tag{3.61}$$

$$\left.\begin{aligned} &\operatorname{Re}\Big\{\overline{t'}\,[-\varkappa\varphi_0(t) + t\overline{\varphi_0'(t)} + \overline{\psi_0(t)}] + \\ &\qquad + \int_{s_{j0}}^{s} \overline{t''}\,[\varkappa\varphi_0(t) - t\overline{\varphi_0'(t)} - \overline{\psi_0(t)}]\,ds\Big\} = 0, \\ &\operatorname{Re}\{\overline{t'}\,[\varphi_0(t) + t\overline{\varphi_0'(t)} + \overline{\psi_0(t)}]\} = 0 \text{ on } L_k \\ &\qquad (k = r_j + 1, \ldots, r_j + \rho_j), \end{aligned}\right\} \tag{3.62}$$

$$-\varkappa\varphi_0(t) + t\overline{\varphi_0'(t)} + \overline{\psi_0(t)} = 0 \text{ on } L_k\ (k = n_2 + 1, \ldots, m), \tag{3.63}$$

$$-\varkappa\varphi_0(t) + t\overline{\varphi_0'(t)} + \overline{\psi_0(t)} + \nu_k \operatorname{Re} \int_{L_{k0}} \overline{t''}\,[\varkappa\varphi_0(t) - t\overline{\varphi_0'(t)} - \overline{\psi_0(t)}]\,ds = 0$$
$$\text{on } L_k\ (k = m + 1, \ldots, n_2 + n_3). \tag{3.64}$$

Substituting $\varphi_0(z)$ and $\psi_0(z)$ in formula (3.20) and taking into account the homogeneous conditions (3.61) to (3.64), we obtain (remembering that $\varkappa > 1$)

$$\varphi_0(z) = i\varepsilon z + \delta, \qquad \psi_0(z) = \delta_1, \tag{3.65}$$

where ε is an arbitrary real constant, and δ, δ_1 are arbitrary complex constants. Substituting in turn expressions (3.65) in any one of the conditions (3.63), (3.64) or in the first condition of (3.62), we find:* $\varepsilon = 0$. After this any of the equalities (3.61), (3.62) gives, as is easily seen, $\delta_1 = -\overline{\delta}$. But then (D. I. Sherman [122]) $\omega_0(t) = \omega_1^{(0)}(t) + i\omega_2^{(0)}(t) = 0$ everywhere on L, and consequently Eqs. (3.60) have no solution in the class H_ε^*.

After fixing the constants C_k $(k = 1, \ldots, n_3)$ on the right-hand side of equality (3.29) in an arbitrary way, we shall try to satisfy the conditions for solvability of system (3.29) in the class H_ε^* by selecting α_j $(j = 1, \ldots, n_1 + n_3)$. These conditions are of the form (N. P. Vekua [139], p. 144)

$$\int_L [F(t) + \beta(t)]\,\tau^{(k)}(t)\,ds = 0 \qquad (k = 1, \ldots, l), \tag{3.66}$$

where $\tau^{(k)}(t) = (\tau_1^{(k)}(t), \tau_2^{(k)}(t))$ are completely specified vectors, the number l of these conditions being determined by the index $\varkappa_0$

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* In the case when either $L^{(1)}$ or $L^{(3)}$ is absent we shall assume that $L^{(2)}$ is not a set of parallel straight segments. Moreover, when $L^{(1)}$ is absent some insignificant modifications are to be introduced into subsequent considerations (see Remark 3 in the author's paper [37]).

introduced above as follows (ibid., p. 145):

$$l = -\varkappa_0 = n_1 + n_3. \tag{3.67}$$

Substituting the expressions for $\beta(t)$ from (3.31) in (3.66), we have

$$\sum_{j=1}^{n_1+n_3} \gamma_{k,j}\alpha_j = f_k^* \qquad (k = 1, \ldots, n_1 + n_3), \tag{3.68}$$

where $\gamma_{k,j}$ are defined real constants independent of $F(t)$ and C_s, and f_k^* are also real constants depending on the latter and vanishing when $F(t) \equiv 0$, $C_s = 0$ $(s = 1, \ldots, n_3)$. Relations (3.68) give a system of linear algebraic equations for the determination of the constants α_j.

Let us prove that the determinant of system (3.68) is different from zero. Let α_j^0 be a solution of the homogeneous system corresponding to (3.68).

It is obvious that the system of integral equations resulting from (3.29) when $F(t) \equiv 0$, $C_s = 0$, $\alpha_j = \alpha_j^{(0)}$ $(s = 1, \ldots, n_3;\ j = 1, \ldots, n_1 + n_3)$ has a definite solution $(\omega_1^{(0)}, \omega_2^{(0)})$ of the class H^*. For the potentials $\varphi_0(z)$, $\psi_0(z)$ corresponding to this solution we have on the boundary of the region

$$\varphi_0(t) + t\overline{\varphi_0'(t)} + \overline{\psi_0(t)} = 0 \text{ on } L^{(1)}, \tag{3.69}$$

$$\left.\begin{aligned} &\operatorname{Re}\Big\{\bar t'\,[-\varkappa\varphi_0(t) + t\overline{\varphi_0'(t)} + \overline{\psi_0(t)}] + \\ &\qquad + \int_{s_{j0}}^{s} \bar t''\,[\varkappa\varphi_0(t) - t\overline{\varphi_0'(t)} - \overline{\psi_0(t)}]\,ds\Big\} = \alpha_j^{(0)}, \\ &\operatorname{Re}\{\bar t'\,[\varphi_0(t) + t\overline{\varphi_0'(t)} + \overline{\psi_0(t)}]\} = 0 \ \text{ on } L_k \ (k = r_j + \\ &\qquad + 1, \ldots, r_j + \rho_j; \qquad j = 1, \ldots, n), \end{aligned}\right\} \tag{3.70}$$

$$-\varkappa\varphi_0(t) + t\overline{\varphi_0'(t)} + \overline{\psi_0(t)} = \alpha_{k+n-n_2}^{(0)} + i\alpha_{k+n_1-n_2}^{(0)} \text{ on } L_k$$
$$(k = n_2 + 1, \ldots, m), \tag{3.71}$$

$$-\varkappa\varphi_0(t) + t\overline{\varphi_0'(t)} + \overline{\psi_0(t)} + \nu_k \operatorname{Re} \int_{L_{k0}} \bar t''\,[\varkappa\varphi_0(t) - t\overline{\varphi_0'(t)} - \overline{\psi_0(t)}]\,ds =$$
$$= \lambda_k\alpha_{k+n_1-n_2}^{(0)} + \nu_k\alpha_j \quad \text{on} \quad L_k$$
$$(k = m + p_j + 1, \ldots, m + p_j + \pi_j; \qquad j = 1, \ldots, n). \tag{3.72}$$

By applying identity (3.20) to the functions $\varphi_0(z)$, $\psi_0(z)$ and using the preceding conditions, we find as before: $\varphi_0(z) = i\varepsilon z + \delta$, $\psi_0(z) = \delta_1$. After this, by a reasoning similar to that used above, we obtain $\varphi_0(z) = 0$, $\psi_0(z) = 0$ in the region S, a result which means, by virtue of the same equalities (3.70) to (3.72) and the condition $\lambda_k \neq 0$, that all $\alpha_j^{(0)} = 0$ $(j = 1, \ldots, n_1 + n_3)$ and this proves our statement.

From the foregoing it follows that if the constants α_j on the right-hand side of equality (3.29) are replaced by their values from relation (3.68), we obtain a system of integral equations solvable in the class H^* for any choice of the values of C_k.

Let (ω_1, ω_2) be a solution (of the class H_ε^*) to this system,* and $\varphi(z)$, $\psi(z)$ the corresponding holomorphic functions. Then a function $u(x, y)$, continuous together with its first derivatives in $S + L$, defined by the formula

$$u(x, y) = \bar{z}\varphi(z) + z\overline{\varphi(z)} + \chi(z) + \overline{\chi(z)}$$

satisfies the boundary conditions

$$\left.\begin{array}{ll} \dfrac{du}{dn} = -\dfrac{dw_0}{dn} \text{ on } L^{(1)}, & G(u) = -G(w_0) \text{ on } L^{(2)}, \\ G(u) = -G(w_0), & H(u) = -H(w_0) \text{ on } L^{(3)}, \\ u = -w_0 + \gamma_j \text{ on } L_j & (j = N+1, \ldots, N+n_3), \end{array}\right\} \qquad (3.73)$$

where γ_j are some (real) constants, and $L_{N+1}, \ldots, L_{N+n_3}$ are unclosed arcs which remain after all arcs of the set $L^{(3)}$ have been removed from L.

We now try to choose the constants C_k so as to satisfy the required boundary conditions for the unknown function u to within the same term on all L_j $(j = N + 1, \ldots, N + n_3)$.

It is easily verified that the last requirement $(\gamma_{N+1} = \ldots = \gamma_{N+n_3})$, together with condition (3.8), gives a system of linear equations for determining C_k

$$\sum_{j=1}^{n_3} \beta_{k,j} C_j = f_k \qquad (k = 1, \ldots, n_3), \qquad (3.74)$$

where $\beta_{k,j}$ are defined constants independent of $F(t)$, and f_k are also constants depending on $F(t)$ and vanishing when $F(t) \equiv 0$.

Let us prove that system (3.74) uniquely determines the constants C_k. Let $C_k^{(0)}$ be a solution of the homogeneous system, and $\varphi_0(z)$, $\psi_0(z)$ holomorphic functions in S defined by means of the solution (of the class H_ε^*) of the integral equations (3.29) with $F(t) \equiv 0$, $C_k = C_k^0$ $(k = 1, \ldots, n_3)$. Then the biharmonic function

$$u_0(x, y) = \bar{z}\varphi_0(z) + z\overline{\varphi_0(z)} + \chi_0'(z) + \overline{\chi_0(z)},$$

where $\chi_0(z)$ is a primitive of $\psi_0(z)$, satisfies the boundary conditions

$$\left.\begin{array}{ll} \dfrac{du_0}{dn} = 0 \text{ on } L^{(1)}, & G(u_0) = 0 \text{ on } L^{(2)}, \\ G(u_0) = H(u_0) = 0 \text{ on } L^{(3)}, & \\ u_0 = \gamma \text{ on } L_k & (k = N+1, \ldots, N+n_3); \end{array}\right\} \qquad (3.75)$$

here γ is some (real) constant.

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* It is obvious that the function $\omega(t) = \omega_1(t) + i\omega_2(t)$ will depend linearly on the undetermined constants C_k $(k = 1, \ldots, n_3)$.

By applying formulas (3.19a), (3.20) to the function $u_0 - \gamma$ and using equalities (3.75), we obtain

$$\varphi_0(z) = i\varepsilon z + \delta, \qquad \psi_0(z) = \delta_1.$$

But by virtue of condition (3.8) $\varepsilon = 0$, consequently $\varphi_0(z) = \delta$ and $\psi_0(z) = \delta_1$.

According to the definition of the functions $\varphi_0(z)$, $\psi_0(z)$ we have on the free part of the boundary L

$$\frac{d}{dt}[-\varkappa\varphi_0(t) + t\overline{\varphi_0'(t)} + \overline{\psi_0(t)}] = iC_{k-n_2} \text{ on } L_k$$

$$(k = n_2 + 1, \ldots, n_2 + n_3).$$

From this it follows that all $C_j^{(0)} = 0$ $(j = 1, \ldots, n_3)$ and our statement is proved.

If the solution of system (3.74) is now substituted into the right side of system (3.29) and holomorphic functions $\varphi(z)$, $\psi(z)$ are formed with the aid of the unique solution (of the class H_ε^*) of the resulting integral equations according to formulas (3.21), the corresponding biharmonic function $u(x, y)$ will satisfy conditions (3.75) with the same γ_j's. By subtracting their common value from u, we obtain the solution of our problem.

4. Continuation

In the case when any one of the sets $L^{(1)}$, $L^{(2)}$, $L^{(3)}$ is absent, the integral equations of the preceding section are appreciably simplified and the problem can be solved in terms of Fredholm equations without recourse to the techniques of the theory of singular equations.* As an illustration, consider the less favourable case when a part of the plate edge is clamped and the remainder is simply supported.

Thus, let the contour of the plate L be made up of $2n$ arcs L_k with ends a_k and a_{k+1} $(k = 1, \ldots, 2n;\ a_{2n+1} = a_1)$. Assume

$$L^{(1)} = \sum_{k=1}^{n} L_{2k-1}, \qquad L^{(2)} = \sum_{k=1}^{n} L_{2k}.$$

Consider the plate to be simply supported on the part $L^{(1)}$ and clamped on $L^{(2)}$.

According to formulas (3.9) to (3.11) we have the following boundary conditions:

$$\operatorname{Re}\left\{(1+\nu)\varphi'(t) + \frac{1}{2}(1-\nu)[t\overline{\varphi''(t)} + \overline{\psi'(t)}]e^{-2i\theta}\right\} = -\frac{1}{4}G(w_0), \quad (4.1)$$

$$\operatorname{Re}\{\overline{t'}[\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)}]\} = -\frac{1}{2}\frac{dw_0}{ds} \text{ on } L^{(1)} \qquad (4.2)$$

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* See the author's paper [36].

and

$$\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}=f(t) \text{ on } L^{(2)}, \tag{4.3}$$

$$f(t)=-\frac{1}{2}\left(\frac{\partial w_0}{\partial x}+\frac{\partial w_0}{\partial y}\right).$$

By integrating relation (4.1) as before with respect to s along each arc L_{2k-1} and adding to equality (4.2) multiplied by $\frac{1}{2}(1-\nu)$, we reduce the contour conditions on the simply supported part to the form

$$\left.\begin{aligned} &\operatorname{Re}\left\{2\bar{t}'\varphi(t)+\frac{1}{\lambda}\int\limits_{a_{2k-1}}^{t}\bar{t}_1''\bar{t}_1'\left[(1-2\lambda)\varphi(t_1)+t_1\overline{\varphi'(t_1)}+\right.\right.\\ &\qquad\qquad\qquad\qquad\left.+\overline{\psi(t_1)}\right]dt_1=g_1(t)+C_{2k-1},\\ &\operatorname{Re}\{\bar{t}'[\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}]\}=g_2(t)\\ &\qquad\qquad\qquad\qquad \text{on } L_{2k-1} \qquad (k=1, \ldots, n), \end{aligned}\right\} \tag{4.4}$$

where C_{2k-1} are undetermined real constants,

$$g_1(t)=-\frac{1}{4}\left[\frac{2}{\lambda}\frac{dw_0}{ds}+\int\limits_{l_{2k-1}}^{s}G(w_0)\,ds_1\right], \qquad g_2(t)=-\frac{1}{2}\frac{dw_0}{ds};$$

here $\lambda=\dfrac{2}{1-\nu}$.

This time the functions $\varphi(z)$ and $\psi(z)$ will be taken in the form of (2.10) when $k=1$

$$\varphi(z)=\frac{1}{2\pi i}\int\limits_{L}\frac{\omega(t)\,dt}{t-z}, \tag{4.5}$$

$$\psi(z)=\frac{1}{2\pi i}\int\limits_{L}\frac{\overline{\omega(t)}\,dt}{t-z}+\frac{1}{2\pi i}\int\limits_{L}\frac{\omega(t)\,\overline{dt}}{t-z}-\frac{1}{2\pi i}\int\limits_{L}\frac{\bar{t}\omega(t)\,dt}{(t-z)^2}, \tag{4.6}$$

where $\omega(t)$ is a new unknown function of the point of the contour L, Hölder continuous on L.

Insert expressions (4.5), (4.6) in the contour conditions (4.3), (4.4), and to the left-hand sides of the equalities obtained from condition (4.3) and the second condition of (4.4) add, respectively, the expressions

$$\frac{b}{t}+\frac{\bar{b}}{\bar{t}}\left(1-\frac{t}{\bar{t}}\right), \qquad \operatorname{Re}\left\{\bar{t}'\left[\frac{b}{t}+\frac{\bar{b}}{\bar{t}}\left(1-\frac{t}{\bar{t}}\right)\right]\right\}, \tag{4.7}$$

where b is a pure imaginary constant related to $\omega(t)$ by the equation

$$b=\frac{1}{2\pi i}\int\limits_{L}\left\{\frac{\omega(t)}{t^2}\,dt+\frac{\overline{\omega(t)}}{\bar{t}^2}\,\overline{dt}\right\}.$$

Then, for the determination of $\omega(t)$ we obtain the equations

$$\frac{1}{2}\left[\bar{t}_0'\omega(t_0)+t_0'\overline{\omega(t_0)}\right]+\frac{1}{2\pi i}\int\limits_{L_{2k-1}}\frac{\bar{t}'\omega(t)-t'\overline{\omega(t)}}{t-t_0}\,dt+$$

$$+{\sum_1^{2n}}_k'\frac{1}{2\pi i}\int\limits_{L_p}\frac{\bar{t}'\omega(t)-t'\overline{\omega(t)}}{t-t_0}\,dt-$$

$$-\operatorname{Re}\left\{\int\limits_{a_{2k-1}}^{t_0}\bar{t}''\bar{t}'\left[\frac{\lambda-1}{\lambda}\,\omega(t)+\frac{1}{\pi i}\int\limits_L\frac{\omega(t_1)\,dt_1}{t_1-t}-\frac{K(t)}{\lambda}\right]dt\right\}-K_1(t_0)=$$

$$=g_1(t_0)+C_{2k-1}, \qquad (4.8'$$

$$\frac{1}{2}\left[\bar{t}_0'\omega(t_0)+t_0'\overline{\omega(t_0)}\right]+\operatorname{Re}\left\{\bar{\bar{t}}{}_0'K^*(t_0)\right\}=$$

$$=g_2(t_0)\ \text{ on }\ L_{2k-1}\qquad (k=1,\ \ldots,\ n), \qquad (4.8'')$$

$$\omega(t_0)+K^*(t_0)=h(t_0)\ \text{ on }\ L^{(2)}, \qquad (4.9)$$

where

$$\left.\begin{aligned}
K(t_0)&=\frac{1}{2\pi i}\int\limits_L\omega(t)\,d\ln\frac{t-t_0}{\bar{t}-\bar{t}_0}-\frac{1}{2\pi i}\int\limits_L\overline{\omega(t)}\,d\,\frac{t-t_0}{\bar{t}-\bar{t}_0},\\
K^*(t_0)&=K(t_0)+\frac{b}{t_0}+\frac{\bar{b}}{\bar{t}_0}\left(1-\frac{t_0}{\bar{t}_0}\right),\\
K_1(t_0)&=\frac{1}{2\pi i}\int\limits_L\frac{(\bar{t}'-\bar{t}_0')\,\omega(t)-(t'-t_0')\,\overline{\omega(t)}}{t-t_0}\,dt-\\
&\qquad-\frac{t_0'}{2\pi i}\int\limits_L\overline{\omega(t)}\,d\ln\frac{t-t_0}{\bar{t}-\bar{t}_0}.
\end{aligned}\right\} \qquad (4.10)$$

It is easy to see that the set of equalities (4.8), (4.9) represents a system of singular integral equations with discontinuous coefficients of the normal type whose right-hand side contains, in addition to the known functions, the constants C_{2k-1} ($k = 1, 2, \ldots, n$) not yet determined.

It is readily proved (see A. I. Kalandiya [36]) that if these equations admit a solution (possessing a certain property of smoothness on L), then it is necessary that $b = 0$, and consequently the functions $\varphi(z)$ and $\psi(z)$ defined through this solution will exactly satisfy the required boundary conditions.

We proceed to the transformation of equalities (4.8).* By subtracting the second equality from the first, we obtain

$$\frac{1}{2\pi i}\int\limits_{L_{2k-1}} \frac{\bar{t}'\omega(t)-t'\overline{\omega(t)}}{t-t_0}\,dt=\Omega(t_0)+C_{2k-1} \text{ on } L_{2k-1} \qquad (k=1,\ldots,n), \qquad (4.11)$$

where C_{2k-1} are constants which are to be determined,

$$\Omega(t_0)=-\sum_{1}^{2n}{}'_{2k-1}\frac{1}{2\pi i}\int\limits_{L_p}\frac{\bar{t}'\omega(t)-t'\overline{\omega(t)}}{t-t_0}\,dt+$$

$$+\operatorname{Re}\left\{\bar{t}_0'K^*(t_0)+\int\limits_{a_{2k-1}}^{t_0}\bar{t}''t\left[\frac{\lambda-1}{\lambda}\omega(t)+\frac{1}{\pi i}\int\limits_{L}\frac{\omega(t_1)dt_1}{t_1-t}-\frac{K(t)}{\lambda}\right]dt\right\}+$$

$$+K_1(t_0)+g_1(t_0)-g_2(t_0). \qquad (4.12)$$

Solving Eq. (4.11) as if the right-hand side were a given function (apart from an additive constant) and taking into account that the unknown function $\omega(t)$ must satisfy the Hölder condition on L, we obtain

$$\frac{1}{2}\left[\bar{t}_0'\omega(t_0)-t_0'\overline{\omega(t_0)}\right]=\frac{R_k(t_0)}{2\pi i}\int\limits_{L_{2k-1}}\frac{\Omega(t)\,dt}{R_k(t)(t-t_0)} \quad (t_0 \text{ on } L_{2k-1}; \quad k=1,\ldots,n), \qquad (4.13)$$

where $R_k(t)=\sqrt{(t-a_{2k-1})(t-a_{2k})}$.

The constants C_{2k-1} are then determined from the requirement for continuity of $\omega(t)$ on L_{2k-1} in the form of the following functional:

$$C_{2k-1}=\frac{1}{\pi i}\int\limits_{L_{2k-1}}\frac{\Omega(t)\,dt}{R_k(t)}.$$

We now transform the right-hand side of (4.13). In the first place, using the easily provable equality

$$-\frac{1}{\pi i}\int\limits_{a_{2k-1}}^{t_0}\bar{t}_1''\bar{t}_1'\,dt_1\int\limits_{L}\frac{\omega(t)\,dt}{t-t_1}=\int\limits_{a_{2k-1}}^{t_0}\bar{t}''\bar{t}'\omega(t)\,dt+$$

$$+\frac{\bar{t}_0''\bar{t}_0'}{\pi i}\int\limits_{L}\omega(t)\ln\left(1-\frac{t_0}{t}\right)dt-\frac{1}{\pi i}\int\limits_{L}\omega(t)\,dt\int\limits_{a_{2k-1}}^{t_0}\ln\left(1-\frac{t_1}{t}\right)d(\bar{t}_1''\bar{t}_1'),$$

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* The idea of transforming the singular system (4.8), (4.9) to Fredholm equations, used below, is taken from D. I. Sherman [123] who applied it to the investigation of the fundamental mixed problem of the plane theory of elasticity.

where by $\ln\left(1-\frac{z}{t}\right)$ for a given t is meant a branch vanishing at $z=0$, we can write

$$\int_{a_{2k-1}}^{t_0} \bar{t}''\bar{t}'\left[\frac{\lambda-1}{\lambda}\,\omega(t)+\frac{1}{\pi i}\int_L \frac{\omega(t_1)\,dt_1}{t_1-t}\right]dt=$$

$$=-\frac{1}{\lambda}\int_{a_{2k-1}}^{t_0}\bar{t}''\bar{t}'\,\omega(t)\,dt-\frac{\bar{t}_0''\bar{t}_0'}{\pi i}\int_L \omega(t)\ln\left(1-\frac{t_0}{t}\right)dt+$$

$$+\frac{1}{\pi i}\int_L \omega(t)\,dt\int_{a_{2k-1}}^{t}\ln\left(1-\frac{t_1}{t}\right)d\left(\bar{t}_1''\bar{t}_1'\right). \qquad (4.14)$$

Consider now an expression of the form

$$I_{2k-1}(t_0)=\frac{R_k(t_0)}{\pi i}\int_{L_{2k-1}}\frac{dt_1}{R_k(t_1)\,(t_1-t_0)}\int_{a_{2k-1}}^{t_1}\tau(t)\,dt \quad (t_0,\ t_1 \text{ on } L_{2k-1}),$$

where $\tau(t)$ is a continuous function on L_{2k-1}. In perfect analogy to equality (3.39) we write

$$I_{2k-1}(t_0)=-\int_{a_{2k-1}}^{t_0}\tau(t)\,dt+$$

$$+\frac{R_k(t_0)}{\pi i}\lim_{z\to t_0}\int_{L_{2k-1}}\frac{dt_1}{R_k(t_1)\,(t_1-z)}\int_{a_{2k-1}}^{t_1}\tau(t)\,dt. \qquad (4.15)$$

Further we have

$$\frac{1}{\pi i}\int_{L_{2k-1}}\frac{dt_1}{R_k(t_1)\,(t_1-z)}\int_{a_{2k-1}}^{t_1}\tau(t)\,dt=$$

$$=\frac{1}{\pi i}\int_{L_{2k-1}}\tau(t)\,dt\int_{t}^{a_{2k}}\frac{dt_1}{R_k(t_1)\,(t_1-z)}.$$

The preceding equality can also be written as

$$\frac{1}{\pi i}\int_{L_{2k-1}}\frac{dt_1}{R_k(t_1)\,(t_1-z)}\int_{a_{2k-1}}^{t}\tau(t)\,dt=$$

$$=\frac{1}{\pi i}\int_{L_{2k-1}}\tau(t)\,dt\int_{L_{2k-1}}\frac{\sigma(t_1,t)\,dt_1}{R_k(t_1)\,(t_1-z)}, \qquad (4.16)$$

where $\sigma(t_1, t)$ is a function of two points on L_{2k-1} defined as follows:

$$\sigma(t_1, t) = \begin{cases} 0 \text{ when } s_1 < s, \\ 1 \text{ when } s_1 \geqslant s; \end{cases}$$

here s_1, s denote the arc abscissas of the points t_1 and t, respectively. Substituting expression (4.16) in equality (4.15) and passing to the limit as $z \to t_0$ on L_{2k-1} we have

$$I_{2k-1}(t_0) = \frac{R_k(t_0)}{\pi i} \int\limits_{L_{2k-1}} \sigma(t_0, t)\, dt \int\limits_{L_{2k-1}} \frac{\sigma(t_1, t)\, dt_1}{R_k(t_1)(t_1 - t_0)}. \qquad (4.17)$$

On the basis of formulas (3.41), (4.14) and (4.17) equality (4.13), after substitution of expression (4.12) in it, can be represented as

$$\frac{1}{2}[\bar{t}_0'\omega(t_0) - t_0'\overline{\omega(t_0)}] = \sum_{1}^{2n}{}'_{2k-1} \frac{R_k(t_0)}{2\pi i} \int\limits_{L_p} \frac{\bar{t}'\omega(t) - t'\overline{\omega(t)}}{R_k(t)(t - t_0)}\, dt -$$

$$- \frac{R_k(t_0)}{2\pi i\lambda} \int\limits_{L_{2k-1}} \sigma(t_0, t)\, \bar{t}' [\bar{t}''\omega(t) + t''\overline{\omega(t)}]\, dt \int\limits_{L_{2k-1}} \frac{\sigma(t_1, t)\, dt_1}{R_k(t_1)(t_1 - t_0)} +$$

$$+ \frac{R_k(t_0)}{\pi i} \int\limits_{L_{k2-1}} \frac{K_2(t) + g_1(t) - g_2(t)}{R_k(t)(t - t_0)}\, dt, \qquad (4.18)$$

where

$$K_2(t_0) = \operatorname{Re}\left\{\bar{t}_0' K^*(t_0) - \frac{1}{\lambda} \int\limits_{a_{2k-1}}^{t_0} \bar{t}''\bar{t}' K(t)\, dt - \right.$$

$$- \frac{\bar{t}_0''\bar{t}_0'}{\pi i} \int\limits_{L} \omega(t) \ln\left(1 - \frac{t_0}{t}\right) dt +$$

$$\left. + \frac{1}{\pi i} \int\limits_{L} \omega(t)\, dt \int\limits_{a_{2k-1}}^{t_0} \ln\left(1 - \frac{t_1}{t}\right) d(\bar{t}_1''\bar{t}_1') \right\} + K_1(t_0). \qquad (4.19)$$

Further we have

$$\sum_{p=1}^{2n}{}'_{2k-1} \frac{R_k(t_0)}{2\pi i} \int\limits_{L_p} \frac{\bar{t}'\omega(t) - t'\overline{\omega(t)}}{R_k(t)(t - t_0)}\, dt =$$

$$= \sum_{p=1}^{n}{}'_{k} \frac{R_k(t_0)}{2\pi i} \int\limits_{L_{2p-1}} \frac{\bar{t}'\omega(t) - t'\overline{\omega(t)}}{R_k(t)(t - t_0)}\, dt +$$

$$+ \frac{R_k(t_0)}{2\pi i} \int\limits_{L^{(2)}} \frac{\bar{t}'\omega(t) - t'\overline{\omega(t)}}{R_k(t)(t - t_0)}\, dt. \qquad (4.20)$$

Note that we have equality (4.9) for $\omega(t)$ on $L^{(2)}$. Therefore if expression (4.20) is inserted in equality (4.18), having previously transformed the integrand in the second integral on the right-hand side of (4.20) with the use of (4.9), we obtain an equality which will no longer contain the unknown $\omega(t)$ under the improper integral. This equality, together with the second equality of (4.8), determines the required form of equations on L_{2k-1} ($k = 1, \ldots, n$).

Thus, we have

$$\left.\begin{aligned} &\overline{t_0'}\omega(t_0) - t_0'\overline{\omega(t_0)} = M_1\{\omega(t),\ g(t),\ t_0\},\\ &\overline{t_0'}\omega(t_0) + t_0'\overline{\omega(t_0)} = M_2\{\omega(t),\ g(t),\ t_0\} \text{ on } L_{2k-1}\\ &\qquad\qquad\qquad\qquad\qquad\qquad (k = 1,\ \ldots,\ n), \end{aligned}\right\} \tag{4.21}$$

where

$$M_1\{\omega(t),\ g(t),\ t_0\} = -\frac{R_k(t_0)}{\pi i}\int\limits_{L^{(2)}}\frac{\overline{t}'K^*(t) - t'\overline{K^*(t)}}{R_k(t)(t-t_0)}\,dt +$$

$$+\frac{R_k(t_0)}{\pi i}\int\limits_{L^{(2)}}\frac{\overline{t}'h(t) - t'\overline{h(t)}}{R_k(t)(t-t_0)}\,dt + M_0\{\omega(t),\ g(t),\ t_0\},$$

$$M_0\{\omega(t),\ g(t),\ t_0\} = R_k(t_0)\left\{\sum_{1}^{n}{}'_{k}\,\frac{1}{\pi i}\int\limits_{L_{2p-1}}\frac{\overline{t}'\omega(t) - t'\overline{\omega(t)}}{R_k(t)(t-t_0)}\,dt -\right.$$

$$-\frac{1}{\pi i\lambda}\int\limits_{L_{2k-1}}\sigma(t_0,\ t)\,\overline{t}'\,[\overline{t}''\omega(t) + t''\overline{\omega(t)}]\,dt\int\limits_{L_{2k-1}}\frac{\sigma(t_1,\ t)\,dt_1}{R_k(t_1)(t_1-t_0)} +$$

$$+\frac{2}{\pi i}\int\limits_{L_{2k-1}}\frac{K_2(t) + g_1(t) - g_2(t)}{R_k(t)(t-t_0)}\,dt,$$

$$M_2\{\omega(t),\ g(t),\ t_0\} = -2\,\mathrm{Re}\,\{\overline{t_0'}K^*(t_0)\} + 2g_2(t_0) \qquad (t_0 \text{ on } L_{2k-1}).$$

The set of equalities (4.21) and (4.9) give, for the determination of $\omega(t)$, a system of Fredholm's integral equations equivalent to system (4.8), (4.9) (from the standpoint of finding solutions which satisfy the Hölder condition).

By a reasoning exactly similar to that used in the preceding section it may be shown (see A. I. Kalandiya [36]) that, under our assumptions regarding the smoothness of the right-hand sides of relations (4.21), (4.9) and the contour L, any (absolutely integrable) solution $\omega(t)$ of these integral equations is a continuous function on the curve L which satisfies the Hölder condition and has an absolutely continuous first derivative. In a similar way it is proved (see ibid.)

that the system of equations (4.21), (4.9) has a unique solution* by means of which the solution of our boundary value problem is constructed.

5. Brief Remarks Concerning Other Results

The mixed problem of the plane theory of elasticity (when external forces are given on a part of the boundary and elastic displacements on the remainder) was studied in the work of D. I. Sherman [120, 123]. In the first of these, published as early as 1938, D. I. Sherman succeeded in extending to the case of mixed boundary conditions a statement of N. I. Muskhelishvili according to which the plane problem can be solved in finite form (by quadrature) for singly connected regions that can be mapped onto a circle by means of rational functions. The other work is devoted to a general theoretical study of the problem for arbitrary regions of finite connectedness. Proceeding from the representations of the complex potentials in the form (3.21) or (4.5), (4.6) suggested by himself (incidentally, very good), D. I. Sherman first reduces the problem to a system of singular integral equations, then transforms it into a Fredholm system and gives a complete investigation of the derived integral equations.*

The mixed problem of the bending of a plate when a part of the closed contour bounding it is fixed and the remainder is free was studied by G. F. Mandzhavidze [77]. The author, just as we did in the preceding sections, used the representations of D. I. Sherman and reduced the problem to a singular integral equation (with discontinuous coefficients) containing, along with the unknown complex-valued function $\omega(t)$, its complex conjugate under the improper integral. Further investigation (in contrast to D. I. Sherman) was carried out without reducing the singular equations to Fredholm's, and for that the author had to extend the fundamental propositions of the theory of singular integral equations to the equation of the special kind obtained by him.

A great step forward in further development of mixed problems was the investigations of V. D. Kupradze [62] and G. F. Mandzhavidze [78] relating to plane anisotropic elasticity.

In the case of the anisotropy of a medium the potentials φ and ψ depend, as is known, on different complex variables, z_1 and z_2, and hence in the corresponding linear problem (with discontinuous coefficients) for φ and ψ the boundary values of these functions will be related at different points of the contour L. These points are displaced relative to each other; the function $\alpha(t)$ characterizing the "displacement" of the points of the boundary is assigned in advance and depends on the elastic properties of the material. By using the complex potentials G. F. Mandzhavidze reduces the problem to a system of singular integral equations and gives its investigation by application of the results of N. P. Vekua.

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* The absence of a non-trivial solution for the homogeneous system is ensured by introducing properly chosen additional terms [see (4.7)] into the left-hand sides of the equalities expressing the boundary conditions of the problem. Such a modification of the integral equations was used by D. I. Sherman [122] in the plane problem with given external forces.

* For details reference should be made to the cited works of D. I. Sherman and also the monograph by N. I. Muskhelishvili [93] (Secs. 101 to 103).

Simultaneously and independently, with the use of real elastic potentials, the problem was investigated in the work of V. D. Kupradze [62] (see also V. D. Kupradze and T. V. Burchuladze [63]). In the aforenamed monograph by V. D. Kupradze [62] one may find many other examples of the application of the method of integral equations.

CHAPTER TWO

Application of Muskhelishvili's Methods

With the name of N. I. Muskhelishvili are directly associated the following methods of solving plane problems in elasticity (N. I. Muskhelishvili [93]):

(1) the method of functional equations (the application of Cauchy-type integrals and conformal mapping; Secs. 78 to 85);

(2) the power series method in conjunction with conformal mapping (Secs. 54 to 64);

(3) the method leading to a problem of linear relationship (Hilbert problem) between analytic functions (Secs. 106 to 128a);

(4) general methods leading to Fredholm's integral equations (Secs. 98 to 105).

The methods listed above have long been finding many applications in theoretical and applied problems of mathematical physics. This is particularly true of the methods (1) and (3).

All these methods, except the fourth, have been developed for the expedient solution of the fundamental plane problem and this aim has been completely achieved. For a rather wide class of singly connected regions whose conformal mapping onto a unit circle is effected by rational functions, the solution of the plane problem by application of these methods is constructed by elementary means in closed form.

But the above methods are found basically suitable for the solution of problems in other cases too, namely when the boundary conditions are of a more complex form than those of the plane problem or else when a region of the medium is not singly connected or is not of the standard form. In these cases the methods of N. I. Muskhelishvili often furnish the means of constructing a solution wholly suitable for practical purposes as well as providing, at the same time, a rigorous justification of the algorithm proposed for solving the problem in a given specific case.

The present chapter illustrates the foregoing by three non-trivial examples.*

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* The chapter reproduces with some modifications the content of the author's papers [38, 39].

6. Simply Supported Plate Whose Middle Surface Can Be Mapped onto a Circle by Means of a Polynomial (Method 3)

Denoting the unknown holomorphic functions by $\varphi_1(z)$, $\psi_1(z)$, we write the boundary conditions (4.1), (4.2) of our problem in the form

$$\left.\begin{aligned} &2\operatorname{Re}\{-t'^2[\overline{t}\varphi_1''(t)+\psi_1'(t)]+\lambda_0\varphi_1'(t)\}=h_1(t),\\ &2\operatorname{Re}\{t'[\overline{\varphi_1(t)}+\overline{t}\varphi_1'(t)+\psi_1(t)]\}=\frac{dh_2(t)}{ds}\text{ on }L.\end{aligned}\right\}\qquad(6.1)$$

Here h_1 and h_2 are given (real) functions of the arc length s on L,

$$h_1(t)=-\frac{G(w_0)}{1-\nu},\qquad h_2'(t)=-w_0,\qquad \lambda_0=2\frac{1+\nu}{1-\nu};\qquad(6.2)$$

w_0 is a particular solution of the bending equation, ν is Poisson's ratio.

The function $\omega(\zeta)$ which maps the region S onto a unit circle in the plane of the variable ζ is assigned in the form*

$$z=\omega(\zeta)=a_1\zeta+a_2\zeta^2+\ldots+a_p\zeta^p.\qquad(6.3)$$

Inserting in expression (6.1) the function $t=\omega(\sigma)$, where σ denotes the affix of a point on the unit circle γ, and taking into account that

$$t'=\frac{i\sigma\omega'(\sigma)}{|\omega'(\sigma)|},$$

we have

$$\left.\begin{aligned} &2\operatorname{Re}\{\sigma^2\omega'(\sigma)[\overline{\omega(\sigma)}\Phi'(\sigma)+\omega'(\sigma)\Psi(\sigma)]+\lambda_0\omega'(\sigma)\varphi'(\sigma)\}=f_1(\sigma),\\ &2\operatorname{Re}\left\{i\sigma\omega'(\sigma)\left[\overline{\varphi(\sigma)}+\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\varphi'(\sigma)+\psi(\sigma)\right]\right\}=f_2(\sigma)\text{ on }\gamma,\end{aligned}\right\}\qquad(6.4)$$

where

$$\varphi_1(z)=\varphi(\zeta),\quad \psi_1(z)=\psi(\zeta),\quad \omega'(\zeta)\Phi(\zeta)=\varphi'(\zeta),\quad \omega'(\zeta)\Psi(\zeta)=\psi'(\zeta),\qquad(6.5)$$

$$f_1(\sigma)=|\omega'(\sigma)|^2h_1[\omega(\sigma)],\qquad f_2(\sigma)=|\omega'(\sigma)|\frac{d}{ds}h_2[\omega(\sigma)].\qquad(6.6)$$

Let us introduce, following N. I. Muskhelishvili, sectionally holomorphic functions $\Omega_1(z)$, $\Omega_2(z)$ (with the exception of poles at the points $\zeta=0$, $\zeta=\infty$) defined by the following equalities (cf.

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* For simplicity's sake, we restrict ourselves to the consideration of polynomial mapping. In the same way the problem can be solved in a more general case when $\omega(\zeta)$ is an arbitrary rational function of ζ.

N. I. Muskhelishvili's book [93], p. 492):

$$\Omega_1 = \begin{cases} \zeta^2\omega'(\zeta)\left[\overline{\omega}\left(\frac{1}{\zeta}\right)\Phi'(\zeta) + \omega'(\zeta)\Psi(\zeta)\right] + \lambda_0\overline{\omega}'\left(\frac{1}{\zeta}\right)\varphi'(\zeta) \\ \qquad \text{for } |\zeta| < 1, \\ -\frac{1}{\zeta^2}\overline{\omega}'\left(\frac{1}{\zeta}\right)\left[\omega(\zeta)\overline{\Phi}'\left(\frac{1}{\zeta}\right) + \overline{\omega}'\left(\frac{1}{\zeta}\right)\overline{\Psi}\left(\frac{1}{\zeta}\right)\right] - \\ \qquad -\lambda_0\omega'(\zeta)\overline{\varphi}'\left(\frac{1}{\zeta}\right) \quad \text{for } |\zeta| > 1, \end{cases} \tag{6.7}$$

$$\Omega_2 = \begin{cases} \frac{1}{\zeta}\overline{\omega}'\left(\frac{1}{\zeta}\right)\varphi(\zeta) - \zeta(\overline{\omega})\left(\frac{1}{\zeta}\right)\varphi'(\zeta) - \zeta\omega'(\zeta)\psi(\zeta)] \\ \qquad \text{for } |\zeta| < 1, \\ \zeta\omega'(\zeta)\overline{\varphi}\left(\frac{1}{\zeta}\right) - \frac{1}{\zeta}\omega(\zeta)\overline{\varphi}'\left(\frac{1}{\zeta}\right) - \frac{1}{\zeta}\overline{\omega}'\left(\frac{1}{\zeta}\right)\overline{\psi}\left(\frac{1}{\zeta}\right) \\ \qquad \text{for } |\zeta| > 1, \end{cases} \tag{6.8}$$

with the notation $\overline{F}(\zeta) = \overline{F(\overline{\zeta})}$.

In terms of the functions Ω_1 and Ω_2 the boundary conditions (6.4) take the form

$$\Omega_1^+(\sigma) - \Omega_1^-(\sigma) = f_1(\sigma), \tag{6.9}$$

$$\Omega_2^+(\sigma) - \Omega_2^-(\sigma) = if_2(\sigma). \tag{6.10}$$

By solving the boundary value problems (6.9) and (6.10), we find (N. I. Muskhelishvili [93], Sec. 108)

$$\Omega_1(\zeta) = A_1(\zeta) + R_1(\zeta), \qquad \Omega_2(\zeta) = A_2(\zeta) + R_2(\zeta), \tag{6.11}$$

where

$$A_1(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{f_1(\sigma)\,d\sigma}{\sigma - \zeta}, \qquad A_2(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{f_2(\sigma)\,d\sigma}{\sigma - \zeta}; \tag{6.12}$$

$R_1(\zeta)$ and $R_2(\zeta)$ denote rational functions with undetermined coefficients. From the very definition of $\Omega_1(\zeta)$ and $\Omega_2(\zeta)$, it appears that

$$\overline{\Omega}_1\left(\frac{1}{\zeta}\right) = -\Omega_1(\zeta), \qquad \overline{\Omega}_2\left(\frac{1}{\zeta}\right) = \Omega_2(\zeta).$$

On the basis of these identities and formulas (6.11) we have

$$\overline{R}_1\left(\frac{1}{\zeta}\right) = -R_1(\zeta) - \frac{1}{2\pi i}\int_\gamma \frac{f_1(\sigma)\,d\sigma}{\sigma}, \qquad \overline{R}_2\left(\frac{1}{\zeta}\right) = R_2(\zeta), \tag{6.13}$$

where use has been made of the fact that by virtue of the second equality of (6.6)

$$\int_\gamma \frac{f_2(\sigma)\,d\sigma}{\sigma} = 0.$$

From this it follows that the functions $R_1(\zeta)$ and $R_2(\zeta)$ must be of the form*

$$\left.\begin{aligned} R_1(\zeta) &= b_0 + \frac{b_1}{\zeta} + \ldots + \frac{b_{p-1}}{\zeta^{p-1}} - \bar{b}_1\zeta - \ldots - \bar{b}_{p-1}\zeta^{p-1}, \\ R_2(\zeta) &= \frac{c_1}{\zeta} + \ldots + \frac{c_{p-1}}{\zeta^{p-1}} + \bar{c}_1\zeta + \ldots + \bar{c}_{p-1}\zeta^{p-1} \end{aligned}\right\} \qquad (6.14)$$

with

$$b_0 + \bar{b}_0 = -\frac{1}{2\pi i}\int\limits_{\gamma} \frac{f_1(\sigma)\,d\sigma}{\sigma}. \qquad (6.15)$$

As is seen from formulas (6.7) and (6.8), the constants b_k, c_k are some linear combinations of the first p coefficients in the power series expansion of the function $\varphi(\zeta)$ in the neighbourhood of $\zeta = 0$. These quantities, $\varphi'(0), \ldots, \varphi^{(p)}(0)$, are just the unknowns.

Taking into account expressions (6.7) and (6.8), from formulas (6.11) we find, for $|\zeta| < 1$,

$$\left.\begin{aligned} &\zeta^2\omega'(\zeta)\left[\bar{\omega}\left(\frac{1}{\zeta}\right)\Phi'(\zeta) + \omega'(\zeta)\Psi(\zeta)\right] + \lambda_0\bar{\omega}'\left(\frac{1}{\zeta}\right)\varphi'(\zeta) = \\ &\qquad = A_1(\zeta) + R_1(\zeta), \\ &\frac{1}{\zeta}\bar{\omega}'\left(\frac{1}{\zeta}\right)\varphi(\zeta) - \zeta\bar{\omega}\left(\frac{1}{\zeta}\right)\varphi'(\zeta) - \zeta\omega'(\zeta)\psi(\zeta) = \\ &\qquad = A_2(\zeta) + R_2(\zeta). \end{aligned}\right\} \qquad (6.16)$$

By eliminating the function $\psi(\zeta)$ from these equalities and solving the differential equation for $\varphi(\zeta)$, we have

$$\varphi(\zeta) = C\left[\frac{\zeta^2\omega'(\zeta)}{\bar{\omega}'\left(\frac{1}{\zeta}\right)}\right]^{\lambda} + \left[\frac{\zeta^2\omega'(\zeta)}{\bar{\omega}'\left(\frac{1}{\zeta}\right)}\right]^{\lambda}\int\left[\frac{\zeta^2\omega'(\zeta)}{\bar{\omega}'\left(\frac{1}{\zeta}\right)}\right]^{1-\lambda} F(\zeta)\,d\zeta. \qquad (6.17)$$

Here C is an arbitrary constant,

$$\lambda = \frac{1-\nu}{4}, \qquad F(\zeta) = \lambda[\Omega(\zeta) + H'(\zeta)], \qquad (6.18)$$

$$\left.\begin{aligned} \Omega(\zeta) &= \frac{1}{\zeta^2\omega'(\zeta)}[A_1(\zeta) + R_1(\zeta)], \\ H(\zeta) &= \frac{1}{\zeta\omega'(\zeta)}[A_2(\zeta) + R_2(\zeta)]. \end{aligned}\right\} \qquad (6.19)$$

From the above formulas it is seen that the expression for $\varphi(\zeta)$ contains the constants C, b_k, c_k, not yet determined

After $\varphi(\zeta)$ has been found, the function $\psi(\zeta)$ can be determined, directly from the second equality of (6.16).

Thus, we have only to find the above-mentioned constants.

---

* We set $\varphi(0) =$

For definiteness, suppose that $\lambda\,(p+1)$ is not an integer and that the roots of the polynomial

$$\overline{\omega}'\left(\frac{1}{\zeta}\right)\zeta^{p-1} = \bar{a}_1\zeta^{p-1} + \ldots + p\bar{a}_p \tag{6.20}$$

are simple. Let these roots (located inside γ) be $\zeta_1, \ldots, \zeta_{p-1}$.

From the preceding formulas it is easily verified that the function

$$F_*(\zeta) = \left[\frac{\omega'(\zeta)}{\overline{\omega}'\left(\frac{1}{\zeta}\right)\zeta^{p-1}}\right]^{1-\lambda}\zeta^{p+1}F(\zeta) \tag{6.21}$$

is regular in the neighbourhood of the point $\zeta = 0$. The following power series expansion is therefore valid for small $|\zeta|$:

$$F_*(\zeta) = A_0 + A_1\zeta + \ldots + A_{p-1}\zeta^{p-1} + \ldots.$$

Expression (6.17) is now rewritten in the form (remembering that $\lambda < 1$)

$$\begin{aligned}\varphi(\zeta) = {} & \left[\frac{\omega'(\zeta)}{\overline{\omega}'\left(\frac{1}{\zeta}\right)\zeta^{p-1}}\right]^{\lambda}\zeta^{\lambda(p+1)} \times \\ & \times \left\{\int_0^{\zeta}\left[\left(\frac{\omega'(\zeta)}{\overline{\omega}'\left(\frac{1}{\zeta}\right)\zeta^{p-1}}\right)^{1-\lambda}\zeta^{(1-\lambda)(p+1)}F(\zeta) - \sum_{k=0}^{p-1}A_k\zeta^{-\lambda(p+1)+k}\right]d\zeta + \right. \\ & \left. + \sum_{k=1}^{p}\frac{A_{k-1}}{-\lambda(p+1)+k}\zeta^{-\lambda(p+1)+k}\right\} + C\left[\frac{\zeta^2\omega'(\zeta)}{\overline{\omega}'\left(\frac{1}{\zeta}\right)}\right]^{\lambda}.\end{aligned} \tag{6.22}$$

It is easy to see that the requirement for uniqueness of the above expression in the neighbourhood of $\zeta = 0$ gives $C = 0$, and hence

$$\begin{aligned}\varphi(\zeta) = {} & \left[\frac{\omega'(\zeta)}{\overline{\omega}'\left(\frac{1}{\zeta}\right)\zeta^{p-1}}\right]^{\lambda}\zeta^{\lambda(p+1)} \times \\ & \times \left\{\int_0^{\zeta}\left[\left(\frac{\omega'(\zeta)}{\overline{\omega}'\left(\frac{1}{\zeta}\right)\zeta^{p-1}}\right)^{1-\lambda}\zeta^{(1-\lambda)(p+1)}F(\zeta) - \sum_{k=0}^{p-1}A_k\zeta^{-\lambda(p+1)+k}\right]d\zeta + \right. \\ & \left. + \sum_{k=1}^{p}\frac{A_{k-1}}{-\lambda(p+1)+k}\zeta^{-\lambda(p+1)+k}\right\}.\end{aligned} \tag{6.23}$$

Note. It is obvious that the coefficients $A_0, A_1, \ldots, A_{p-1}$ contain the quantities $\varphi^{(k)}(0)$ $(k = 1, \ldots, p)$, not yet defined. If for the determination of the latter, we express, as in the case of the plane problem (N. I. Muskhelishvili [93], Sec. 84), the condition that $\varphi^{(1)}(0), \ldots, \varphi^{(p)}(0)$ represent the coefficients in the expansion of the right-hand side of expression (6.23), we obtain, as is easily verified, not equations, but the identities $\varphi^{(k)}(0) = \varphi^{(k)}(0)$, where $k = 1, \ldots, p-1$, and one condition (obtained by comparing the coefficients of ζ^p) which coincides with condition (6.15). The fact is that, as verification shows, the coefficients $A_0, A_1, \ldots, A_{p-2}$ do not contain the known quantities at all and represent some linear combinations of $\varphi^{(1)}(0), \ldots, \varphi^{(p-1)}(0)$; the known quantities in the expansion of the right-hand side of expression (6.23) occur in the coefficients of ζ^k beginning only with $k = p$.

Actually, the unknowns $\varphi^{(1)}(0), \ldots, \varphi^{(p-1)}(0)$ must be determined from the requirement that expression (6.23) which represents, in general, a multiple-valued function should be holomorphic inside γ.

We now express the condition that the function $\varphi(\zeta)$ defined by formula (6.23) should remain finite at the points $\zeta_1, \ldots, \zeta_{p-1}$. With $\lambda > 0$ we have

$$\int_0^{\zeta} \left\{ \left[\frac{\omega'(\zeta)}{\overline{\omega}'\left(\frac{1}{\zeta}\right)\zeta^{p-1}} \right]^{1-\lambda} \zeta^{(1-\lambda)(p+1)} F(\zeta) - \sum_{j=0}^{p-1} A_j \zeta^{-\lambda(p+1)+j} \right\} d\zeta +$$

$$+ \sum_{j=1}^{p} \frac{A_{j-1}}{-\lambda(p+1)+j} \zeta_k^{-\lambda(p+1)+j} = 0 \quad (k=1, \ldots, p-1), \qquad (6.24)$$

where the integral is taken over an arbitrary path joining the points 0 and ζ_k and not passing through the other points ζ_j.

It is easily verified that equalities (6.24) express the necessary and sufficient conditions that the function $\varphi(\zeta)$ should be single-valued in the neighbourhood of the points ζ_k and hence holomorphic inside the unit circle.*

Equalities (6.24) together with (6.15) and the condition

$$\operatorname{Im}\left\{\frac{\varphi'(0)}{\omega'(0)}\right\} = 0 \qquad (6.25)$$

give a system of linear equations for the determination of the unknowns $\varphi^{(k)}(0)$ $(k = 1, \ldots, p)$. That this system does have a solution is evident from the uniqueness of solution of our boundary value problem.

In the case when $\lambda(p+1)$ is an integer or when among the roots of the polynomial $\zeta^{(p-1)}\overline{\omega}'\left(\frac{1}{\zeta}\right)$ there are multiple roots, we obtain, by a slight extension of the argument used previously, conditions, analogous to those indicated above, for the determination of the unknowns C, $\varphi'(0), \ldots, \varphi^{(p)}(0)$.

For example, when $\lambda(p+1)$ is not an integer and $\zeta^{p-1}\overline{\omega}'\left(\frac{1}{\zeta}\right)$ has roots $\zeta_1, \ldots, \zeta_r$ of multiplicity $m_1, \ldots, m_r$, respectively, we again have $C = 0$, and conditions (6.24) are replaced by the following:

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* For a proof see the author's paper [38], pp. 297-298.

$$\frac{d^j}{d\zeta^j}\left\{\left[\frac{\zeta^2(\zeta-\zeta_k)^{m_k}\omega'(\zeta)}{\overline{\omega}'\left(\frac{1}{\zeta}\right)}\right]^{1-\lambda} F(\zeta)\right\} = 0 \quad \text{when } \zeta = \zeta_k,$$

$$\int_0^{\zeta_k}\left[\left(\frac{\zeta^2\omega'(\zeta)}{\overline{\omega}'\left(\frac{1}{\zeta}\right)}\right)^{1-\lambda} F(\zeta) - \sum_{j=0}^{p-1} A_j \zeta^{-\lambda(p+1)+j}\right] d\zeta +$$

$$+\sum_{j=1}^{p}\frac{A_{j-1}}{-\lambda(p+1)+j}\zeta_k^{-\lambda(p+1)+j} = 0$$

$$(j = 0, 1, \ldots, m_k - 2; \qquad k = 1, \ldots, r).$$

The foregoing equalities ensuring the holomorphy of expression (6.23) in the circle again give, together with formulas (6.15) and (6.25), a uniquely solvable system of linear algebraic equations for the determination of the unknowns $\varphi^{(k)}(0)$ $(k = 1, \ldots, p)$.

7. Examples

1°. **Infinite plane with an elliptical hole.** We shall consider an infinite plate with an elliptical hole, which is bent by forces normal to the middle surface and bending moments applied along the edge of the hole. For simplicity, suppose that the set of all external forces applied along the contour of the hole is statically equivalent to zero and, further, that the stresses vanish at infinity. In this case the analytic functions $\varphi_1(z)$, $\psi_1(z)$ are holomorphic in the entire region of the plate, including the point at infinity (see S. G. Lekhnitskii [73]).

These functions may be subjected at infinity to the condition

$$\varphi_1(\infty) = 0 \quad \text{or} \quad \psi_1(\infty) = 0.$$

In addition, because of the single-valuedness of the deflection it is necessary that the function $\psi_1(z)$ should satisfy at infinity the condition

$$\operatorname{Im}\{\lim_{z\to\infty} z[\psi_1(z) - \psi_1(\infty)]\} = 0. \tag{7.1}$$

The problem consists in determining the deflections of the points of the middle surface when the deflections and bending moments are directly given on the boundary of the region (Problem III, Sec. 1).*

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* The solution of Problems II and I when the bending moments and bending forces or the deflections and slopes of the bent middle surface are given on the boundary of the region, respectively, for a number of special cases (for example, for a circular ring, for the outside of an ellipse as well as for a more general shape of holes in the form of a curvilinear polygon) may be found in the work of S. G. Lekhnitskii [72], G. N. Savin [109] and M. M. Fridman [23, 24].

The mapping function in our case is of the form

$$z=\omega(\zeta)=R\left(\frac{1}{\zeta}+m\zeta\right),\qquad R>0,\quad 0<m<1.$$

Hence

$$\omega'(\zeta)=-\frac{R}{\zeta^2}(1-m\zeta^2),\qquad \overline{\omega}\left(\frac{1}{\zeta}\right)=R\left(\zeta+\frac{m}{\zeta}\right),$$

$$\overline{\omega}'\left(\frac{1}{\zeta}\right)=R\,(m-\zeta^2).$$

Since, according to the foregoing, $\varphi(\zeta)$, $\psi(\zeta)$ are holomorphic functions in the circle $|\zeta|<1$, from the preceding formulas and formula (6.5) we conclude that, for small $|\zeta|$,

$$\Phi(\zeta)=O(\zeta^2),\qquad \Psi(\zeta)=O(\zeta^2).$$

Based on this estimate, it is easy to show that the functions $\Omega_1(\zeta)$, $\Omega_2(\zeta)$ defined by equalities (6.7), (6.8) will be bounded near $\zeta=0$, provided the following condition is fulfilled:

$$m\varphi(0)+\psi(0)=0. \tag{7.2}$$

In view of the fact that $m<1$, this condition may always be considered to be satisfied (cf. N. I. Muskhelishvili's work [93], p. 501).

In accordance with the above-said, equalities (6.16) become

$$\left.\begin{aligned}&\zeta^2\omega'(\zeta)\left[\overline{\omega}\left(\frac{1}{\zeta}\right)\Phi'(\zeta)+\omega'(\zeta)\Psi(\zeta)\right]+\\&\qquad+\lambda_0\overline{\omega}'\left(\frac{1}{\zeta}\right)\varphi'(\zeta)=A_1(\zeta)+b,\\&\frac{1}{\zeta}\overline{\omega}'\left(\frac{1}{\zeta}\right)-\zeta\overline{\omega}\left(\frac{1}{\zeta}\right)\varphi'(\zeta)-\zeta\omega'(\zeta)\psi(\zeta)=A_2(\zeta)+c.\end{aligned}\right\} \tag{7.3}$$

Here b, c are constants, not yet determined; the functions $A_1(\zeta)$ and $A_2(\zeta)$ are defined by formula (6.12), where f_1 and f_2 are given by formulas (6.6), with $-D(1-\nu)h_1(s)$ and $h_2(s)$ denoting this time the given values of the bending moment and deflection, respectively.

In our case conditions (6.13) give

$$b+\overline{b}=-\frac{1}{2\pi i}\int\limits_{\gamma}\frac{f_1(\sigma)\,d\sigma}{\sigma},\qquad c=c. \tag{7.4}$$

It can readily be verified that the second of the above conditions entails the validity of equality (7.1) ensuring the single-valuedness of the deflection in the region under consideration.

As the simplest computations show, formula (6.17) yields

$$\varphi(\zeta)=C\left[\frac{1-m\zeta^2}{\zeta^2-m}\right]^{\lambda}+\left[\frac{1-m\zeta^2}{\zeta^2-m}\right]^{\lambda}\int\limits_0^{\zeta}\left[\frac{1-m\zeta^2}{\zeta^2-m}\right]^{1-\lambda}F(\zeta)\,d\zeta, \quad (7.5)$$

where

$$F(\zeta)=-\frac{\lambda}{R\,(1-m\zeta^2)}\Big[A_1(\zeta)+\zeta A_2'(\zeta)+ \\ +\frac{1+m\zeta^2}{1-m\zeta^2}A_2(\zeta)+b+(1+2m\zeta^2)\,c\Big]. \quad (7.6)$$

The function $\bar{\omega}'\left(\frac{1}{\zeta}\right)$ has (inside γ) two roots $\zeta=\pm\sqrt{m}$. Accordingly, conditions (6.24) in our case become

$$\int\limits_0^{\sqrt{m}}\left[\frac{1-m\zeta^2}{\zeta^2-m}\right]^{1-\lambda}F(\zeta)\,d\zeta+C=0, \quad \int\limits_0^{-\sqrt{m}}\left[\frac{1-m\zeta^2}{\zeta^2-m}\right]^{1-\lambda}F(\zeta)\,d\zeta+C=0.$$

2°. **Infinite plane with a circular hole.** As to the nature of the external load and the state of stress at infinity we make the same assumptions as in the preceding example.

Applying this time the transformation $z=R\zeta$, which maps the region under consideration onto the outside of a unit circle, and following the procedure indicated above, we obtain (see N. I. Muskhelishvili [93], pp. 485-487)

$$\varphi(\zeta)=\lambda\zeta^{2\lambda}\int\limits_{\infty}^{\zeta}[RA_1(\zeta)+A_2(\zeta)-\zeta A_2'(\zeta)]\,\zeta^{-2\lambda}\,d\zeta,$$

$$\psi(\zeta)=\frac{\varphi(\zeta)}{\zeta^2}-\frac{\varphi'(\zeta)}{\zeta}+\frac{A_2(\zeta)}{\zeta}+\frac{c}{\zeta},$$

where

$$c=\frac{R}{2\pi i}\int\limits_{\gamma}\frac{f_1(\sigma)\,d\sigma}{\sigma}.$$

8. Plate with Simply Supported Edge in the Form of a Solid Ellipse (Method 1)

We shall consider the problem of the equilibrium of an elliptical plate simply supported along the edge and bent by a normal load of intensity q distributed over the middle surface. For convenience, suppose that the centre of the ellipse is located at the point $z=0$. Let its major and minor semi-axes be denoted by a and b, respectively.*

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* The fundamental plane problem for an ellipse was solved by N. I. Muskhelishvili [93] and D. I. Sherman [124].

1°. Reduction to an infinite system of linear algebraic equations. In the case under consideration, the boundary conditions (6.1) of our problem can be represented in the form of the following single complex equality (Z. I. Khalilov [58], p. 410):

$$\lambda\left[\varphi_1'(t)+\overline{\varphi_1'(t)}\right]+\overline{t''}\left[\varphi_1(t)+t\overline{\varphi_1'(t)}+\overline{\psi_1(t)}\right]=f_1(t). \tag{8.1}$$

Here

$$f_1(t)=\frac{1}{2}\left[h_1(t)+\frac{d^2h_2(t)}{ds^2}-\frac{i}{\rho_0}\,\frac{dh_2(t)}{ds}\right], \quad \lambda=\frac{2}{1-\nu}, \quad t''=\frac{d^2t}{ds^2} \tag{8.2}$$

with ρ_0 as the radius of curvature of the ellipse at the point t_0. In the following discussion it is assumed that $f_1(t)$ has the second derivative with respect to the arc length s which satisfies the Hölder condition with exponent 1 on L.

The solution of problem (8.1) will be sought in the form of (4.5), (4.6):

$$\varphi_1(z)=\frac{1}{2\pi i}\int_L\frac{\omega_1(t)\,dt}{t-z}, \quad \psi_1(z)=\frac{1}{2\pi i}\int_L\frac{\overline{\omega_1(t)}\,dt}{t-z}-\frac{1}{2\pi i}\int_L\frac{\bar{t}\omega_1'(t)\,dt}{t-z}, \tag{8.3}$$

where $\omega_1(t)$ is the unknown function of the point of the contour L.

Substituting expressions (8.3) in the boundary condition (8.1), we have

$$2\lambda\,\mathrm{Re}\left\{\lim_{z\to t_0}\frac{1}{2\pi i}\int_L\frac{\omega_1'(t)\,dt}{t-z}\right\}+$$
$$+\overline{t_0''}\left[\omega_1(t_0)+\frac{1}{2\pi i}\int_L\omega_1(t)\,d\ln\frac{t-t_0}{\bar{t}-\bar{t}_0}-\frac{1}{2\pi i}\int_L\overline{\omega_1(t)}\,d\frac{t-t_0}{\bar{t}-\bar{t}_0}\right]=f_1(t_0). \tag{8.4}$$

Introduce the substitution

$$z=R\left(\zeta+\frac{1}{\zeta}\right), \qquad 2R=\sqrt{a^2-b^2}. \tag{8.5}$$

The above relation transforms ellipses confocal with the given one onto concentric circles with centre at the point $\zeta=0$. To the given ellipse there corresponds a circumference γ of radius ρ defined by the formula

$$\rho=\frac{a+\sqrt{a^2-4R^2}}{2R}. \tag{8.6}$$

To the unit circle in the ζ plane corresponds the segment $[-2R, 2R]$.

Substituting expression (8.5) in relation (8.4) and noting that

$$t=R\left(\sigma+\frac{1}{\sigma}\right), \qquad \overline{t''}=\frac{(\rho^4-1)\,\sigma^3}{R\,(\sigma^2-1)^2\,(\sigma^2-\rho^4)}\,.$$

where σ denotes the affix of a point on γ as before, we obtain

$$\frac{2\lambda}{R}\operatorname{Re}\left\{\lim_{\zeta\to\sigma_0}\frac{\zeta}{\zeta^2-1}\left[\frac{1}{2\pi i}\int_\gamma\frac{\sigma\omega'(\sigma)\,d\sigma}{\sigma-\zeta}-\frac{1}{2\pi i}\int_\gamma\frac{\sigma\omega'(\sigma)\,d\sigma}{\sigma-\frac{1}{\zeta}}\right]\right\}+$$

$$+\frac{(\rho^4-1)\,\sigma_0^3}{\overline{R}\,(\sigma_0^2-1)^2\,(\sigma_0^2-\rho^4)}\left[\omega(\sigma_0)+\frac{1}{2\pi i}\int_\gamma\frac{\omega(\sigma)\,d\sigma}{\sigma-\frac{1}{\sigma_0}}-\right.$$

$$\left.-\frac{1}{2\pi i}\int_\gamma\frac{\omega(\sigma)\,d\sigma}{\sigma-\frac{\rho^4}{\sigma_0}}-\frac{\rho^4-1}{2\pi i}\int_\gamma\frac{\overline{\omega'(\sigma)}\,d\sigma}{\sigma\left[\sigma-\frac{\rho^4}{\sigma_0}\right]}\right]=f(\sigma_0); \tag{8.7}$$

here $\omega(\sigma)=\omega_1(t)$, $f(\sigma)=f_1(t)$.

The function $\omega(\sigma)$ is sought in the form of a Fourier series

$$\omega(\sigma)=\sum_{-\infty}^{\infty}a_k\sigma^k. \tag{8.8}$$

Hence

$$\omega'(\sigma)=\sum_{-\infty}^{\infty}ka_k\sigma^{k-1}, \qquad \overline{\omega'(\sigma)}=\sum_{-\infty}^{\infty}k\overline{a}_k\rho^{2(k-1)}\sigma^{-k+1}. \tag{8.9}$$

Assume further

$$\frac{R\,(\sigma^2-1)^2\,(\sigma^2-\rho^4)}{\sigma^3}f(\sigma)=\sum_{-\infty}^{\infty}A_k\sigma^k. \tag{8.10}$$

Substituting series (8.8) to (8.10) in relation (8.7) and noting that the points ζ, $1/\zeta$, $1/\sigma_0$ are located inside the circle γ and the point ρ^4/σ_0 outside it, with the use of Cauchy's integral formula we obtain (σ_0 is replaced by σ)

$$\lambda\left\{[\sigma^2-(\rho^4+1)+\rho^4\sigma^{-2}]\left[\sum_1^\infty ka_k\sigma^k-\sum_1^\infty ka_k\sigma^{-k}\right]-\right.$$

$$\left.-(\sigma^2-2+\sigma^{-2})\left[\sum_1^\infty k\overline{a}_k\rho^{2(k+1)}\sigma^{-n}-\sum_1^\infty k\overline{a}_k\rho^{-2(k-1)}\sigma^k\right]\right\}+$$

$$+(\rho^4-1)\left\{2a_0+\sum_1^\infty[a_k+a_{-k}\rho^{-4k}+(\rho^4-1)\,k\overline{a}_k\rho^{-2(k+1)}]\,\sigma^k+\right.$$

$$\left.+\sum_1^\infty(a_k+a_{-k})\,\sigma^{-k}\right\}=\sum_{-\infty}^{\infty}A_k\sigma^k.$$

Comparison of the coefficients of σ^k $(k = 0, \pm 1, \ldots)$ gives

$$2(\rho^4 - 1) a_0 + 2\lambda(\rho^4 - 1) a_2 + 2\lambda(\rho^{-2} - \rho^6)\overline{a}_2 = A_0, \tag{8.11$'$}$$

$$\begin{aligned}\lambda(k+2)\,[\rho^4 a_{k+2} + \overline{a}_{k+2}\rho^{-2(k+1)}] - \lambda k\,[(\rho^4+1) a_k + 2\overline{a}_k \rho^{-2(k-1)}] + \\ + (\rho^4 - 1)\,[a_k + a_{-k}\rho^{-4k} + (\rho^4 - 1)\, k\overline{a}_k \rho^{2(k+1)}] + \\ + \lambda(k-2)\,[a_{k-2} + \overline{a}_{k-2}\rho^{-2(k-3)}] = A_k \qquad (k = 1, 2, \ldots);\end{aligned} \tag{8.11$''$}$$

$$\begin{aligned}(\rho^4 - 1) a_{-k} = \lambda(k+2)\,[a_{k+2} + \overline{a}_{k+2}\rho^{2(k+3)}] - \\ - \lambda k\,[(\rho^4+1) a_k + 2\overline{a}_k \rho^{2(k+1)}] + (1 - \rho^4) a_k + \\ + \lambda(k-2)\,[\rho^4 a_{k-2} + \overline{a}_{k-2}\rho^{2(k-2)}] + A_{-k} \qquad (k = 1, 2, \ldots).\end{aligned} \tag{8.12}$$

In these equalities, when $k = 1$, a_{k-2} and $\overline{a}_{k-2}$ must be replaced, respectively, by a_{-k+2} and $\overline{a}_{-k+2}$.

The set of equalities (8.11), (8.12) represents an infinite system of linear algebraic equations in a_k $(k = 0, \pm 1, \ldots)$.*

We shall reduce our system to a somewhat different form, more convenient for investigation. For this purpose, we substitute the values of a_{-k} from relations (8.12) in (8.11) and, in the resulting system for a_k, introduce new unknowns b_k defined by the formula

$$a_0 = b_0, \qquad \lambda k \rho^k a_k = b_k \qquad (k = 1, 2, \ldots). \tag{8.13}$$

After some simplification we have

$$2(\rho^4 - 1) b_0 = (\rho^{-2} - \rho^2) b_2 + (\rho^4 - \rho^{-4})\overline{b}_2 + A_0, \tag{8.14$'$}$$

$$\xi_2 b_2 + \eta_2 \overline{b}_2 = (\rho^2 + \rho^{-10}) b_4 + (1 + \rho^{-8})\overline{b}_4 - B_2, \tag{8.14$''$}$$

$$\begin{aligned}\xi_{2k} b_{2k} + \eta_{2k}\overline{b}_{2k} = \rho^2\,[1 + \rho^{-4(2k+1)}]\, b_{2k+2} + \rho^2\,[1 + \rho^{-4(2k-1)}]\, b_{2k-2} + \\ + (1 + \rho^8)\,\rho^{-4(k+1)}\,[\overline{b}_{2k+2} + \rho^4 \overline{b}_{2k-2}] - B_{2k} \qquad (k = 2, 3, \ldots),\end{aligned} \tag{8.14$'''$}$$

$$\begin{aligned}[(\lambda - 1)(\rho^4 + \rho^{-4}) + 2(1 + 2\lambda)]\,(b_1 + \overline{b}_1) = \\ = \lambda(\rho^2 + \rho^{-6})\,(b_3 + \overline{b}_3) - \lambda B_1,\end{aligned} \tag{8.15$'$}$$

$$\begin{aligned}\xi_{2k+1} b_{2k+1} + \eta_{2k+1}\overline{b}_{2k+1} = \rho^2\,[1 + \rho^{-8(k+1)}]\, b_{2k+3} + \rho^2\,[1 + \rho^{-8k}]\, b_{2k-1} + \\ + (1 + \rho^8)\,\rho^{-2(2k+3)}\,[\overline{b}_{2k+3} + \rho^4 \overline{b}_{2k-1}] - B_{2k+1} \qquad (k = 1, 2, \ldots),\end{aligned} \tag{8.15$''$}$$

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* The procedure used here for reducing the problem to a system of linear equations is nothing other than the method of functional equations (1) in principle, but still outwardly it differs from it. This method would be more properly designated as a combination of Muskhelishvili's methods (1), (2) and (4) (see p. 62).

The available methods of solving plane problems, in spite of their great generality, by no means cover all problems that may be encountered in practice. In cases most important for applications, it is often necessary, therefore, to search for special methods of solution adapted to a particular class of problems. These special methods are most frequently made up of various combinations of the methods already known.

The combined method in question was employed in the above-mentioned work of D. I. Sherman [124] (for further details see N. I. Muskhelishvili [93], p. 599).

where

$$\left.\begin{aligned} &B_k = \rho^k A_k - \rho^{-3k} A_{-k} \qquad (k=1,2,\ldots),\\ &\xi_k = (1+\rho^4)(1+\rho^{-4k}) + \frac{1}{\lambda k}(1-\rho^4)(1-\rho^{-4k}),\\ &\eta_k = \rho^{-2(k+1)}\left[4\rho^4 - \frac{1}{\lambda}(\rho^4-1)^2\right] \qquad (k=2,3,\ldots). \end{aligned}\right\} \tag{8.16}$$

From the first two equalities of (8.15) it is seen that the imaginary part of the coefficient b_1 may be arbitrarily fixed. Furthermore, as the first equality of (8.15) shows, the possibility of solving our problem demands that the quantity $\rho^2 A_1 - \rho^{-2} A_{-1}$ should be real. It can easily be verified that this condition is really fulfilled.

Equalities (8.14), (8.15) represent a set of infinite systems of linear algebraic equations in $b_0, b_2, \ldots$ and $b_1, b_3, \ldots$ (more precisely, in the real and imaginary parts of these unknowns). It can be found without great difficulty that the above-mentioned systems are quasi-regular* for any ρ ($\rho > 1$). It can readily be verified also that for large ρ they will be already completely regular.**

It will later be proved that system (8.14), (8.15) has a unique solution which does give the solution to the boundary value problem under consideration. A method for the approximate solution of this system will also be indicated.

2°. **Investigation of infinite systems of equations.** We shall seek bounded solutions of system (8.14), (8.15). Let us first prove that any such solution b_k has the same order of decreasing with increasing number as the right-hand side of the system.

To begin with, we remark that, according to our assumption regarding the given function $f_1(t)$, for the coefficients in expansion (8.10) we have inequalities of the form

$$\rho^k A_k| < \frac{M}{|k|^3} \qquad (k = \pm 1, \pm 2, \ldots). \tag{8.17}$$

Let now b_k be some bounded solution of system (8.14), (8.15). This system, without a number of its first equations, is rewritten as

$$\rho^k b_{k+2} - (\rho^4+1)\, b_k + \rho^2 b_{k-2} = Q_k$$
$$(k = N, N+1, \ldots, N \geqslant 3), \tag{8.18}$$

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* For infinite systems of equations see, for example, L. V. Kantorovich and V. I. Krylov [57], Chap. 1, Sec. 2.

** As even rough estimates show, for that it is sufficient to take $\rho \geqslant \sqrt{3}$ or, which is the same thing, $a \leqslant 2b$. However, the condition that systems (8.14) and (8.15) should be completely regular will not be fulfilled for every $\rho > 1$; for example, the third equation of system (8.14) does not satisfy this condition even for $\rho = 1.5$.

where use has been made of the notation

$$Q_k = B_k + \frac{1-\rho^4}{\lambda k}\left\{1+\rho^{-4k}\left[\frac{\lambda k(1+\rho^4)}{1-\rho^4}-1\right]\right\} b_k +$$
$$+\rho^{-2(k+1)}\left[4\rho^4 - \frac{1}{\lambda}(\rho^4-1)^2\right]\overline{b}_k - \rho^{-2(2k+1)} b_{k+2} - \rho^{-2(2k-3)} b_{k-2} -$$
$$-(1+\rho^8)\rho^{-2(k+2)}[\overline{b}_{k+2} + \rho^4 \overline{b}_{k-2}] \qquad (k=2, 3, \ldots). \quad (8.19)$$

The set of equalities (8.18) with even numbers (or, which is equivalent, relations (8.14) beginning with $k = 2$) represents, for known Q_{2k}, a second-order linear finite-difference equation with constant coefficients. Solving this equation by the well-known method, as if the right-hand side were a given sequence of numbers, and taking into account that the roots of the characteristic equation here are $\lambda_{1,2} = \rho^{\pm 2}$, we find its general solution in the form (see, for example, A. O. Gel'fond [27])

$$b_{2k} = C_1\rho^{2(k+1)} + C_2\rho^{-2(k+1)} + \sum_{m=N}^{k} \frac{\rho^{2(k-m)} - \rho^{-2(k-m)}}{\rho^4 - 1} Q_{2m}$$
$$(k = N, N+1, \ldots; N \geqslant 2), \quad (8.20)$$

where C_1, C_2 are arbitrary constants. The above equality is rewritten as

$$b_{2k} = \rho^{2(k+1)}\left[C_1 + \frac{\rho^{-2}}{\rho^4-1}\sum_{m=N}^{k}\rho^{-2m}Q_{2m}\right] +$$
$$+\frac{\rho^{-2k}}{1-\rho^4}\sum_{m=N}^{k}\rho^{2m}Q_{2m} + C_2\rho^{-2(k+1)}. \quad (8.21)$$

Since the second sum on the right-hand side, as will be shown later, remains bounded with increasing k, for boundedness of b_{2k} we must set

$$C_1 = \frac{\rho^{-2}}{1-\rho^4}\sum_{m=N}^{\infty}\rho^{-2m}Q_{2m}.$$

Inserting this value for C_1 in formula (8.21), we obtain

$$b_{2k} = \frac{1}{1-\rho^4}\left[\sum_{m=k+1}^{\infty}\rho^{2(k-m)}Q_{2m} + \sum_{m=N}^{k}\rho^{-2(k-m)}Q_{2m}\right] + C_2\rho^{-2(k+1)}. \quad (8.22)$$

Considering the set of equalities (8.18) with odd numbers and reasoning as before, we obtain in a similar way

$$b_{2k+1} = \frac{1}{1-\rho^4}\left[\sum_{m=k+1}^{\infty}\rho^{2(k-m)}Q_{2m+1} + \sum_{m=N}^{k}\rho^{-2(k-m)}Q_{2m+1}\right] + C_2'\rho^{-2(k+1)}$$
$$(k = N, N+1, \ldots; N \geqslant 1). \quad (8.23)$$

Assume further

$$p_k = \sum_{m=k+1}^{\infty} \rho^{2(k-m)} Q_{2m}, \qquad q_k = \sum_{m=N}^{k} \rho^{-2(k-m)} Q_{2m}.$$

We examine the sum

$$q_k = Q_{2k} + \frac{Q_{2k-2}}{\rho^2} + \frac{Q_{2k-4}}{\rho^4} + \dots + \frac{Q_{2N}}{\rho^{2(k-N)}}. \tag{8.24}$$

Since $\{b_k\}$ is a bounded sequence of numbers, on the basis of formulas (8.16) and (8.17) when $k \to \infty$, the right-hand side of expression (8.19) decreases as $1/k$. Substituting expression (8.19) in formula (8.24) and taking into account relations (8.17), we easily conclude that*

$$|q_k| < \frac{M}{k}\left[1 + \frac{k^3}{(k-2)^3 \rho^2} + \frac{k^3}{(k-4)^3 \rho^4} + \dots + \frac{k^3}{N^3 \rho^{2(k-N)}}\right]. \tag{8.25}$$

Note now that the sequence of numbers

$$\alpha_k = \left(\frac{k}{k-1}\right)^m \frac{1}{\rho} + \left(\frac{k}{k-2}\right)^m \frac{1}{\rho^2} + \dots + \left(\frac{k}{2}\right)^m \frac{1}{\rho^{k-2}} + \frac{k^m}{\rho^{k-1}},$$

$$(k = N, N+1, \dots), \tag{8.26}$$

where m is some natural number, is bounded. Indeed, we have

$$|\alpha_k| < \frac{k^m}{\left[\frac{k}{2}\right]^m}\left[\frac{1}{\rho} + \frac{1}{\rho^2} + \dots + \frac{1}{\rho^{[k/2]}}\right] +$$

$$+ \frac{k^m}{\rho^{[k/2]}}\left[\frac{1}{\rho} + \frac{1}{\rho^2} + \dots + \frac{1}{\rho^{k-1-[k/2]}}\right], \tag{8.27}$$

from which the statement is evident.

On this basis, from relation (8.25) we have

$$|q_k| < \frac{M}{k} \qquad (k = N, N+1, \dots). \tag{8.28}$$

Consider further

$$p_k = \frac{Q_{2k+2}}{\rho^2} + \frac{Q_{2k+4}}{\rho^4} + \dots + \frac{Q_{2(k+m)}}{\rho^{2m}}. \tag{8.29}$$

Substituting expression (8.19) in (8.29) and using inequalities (8.17), we immediately find

$$|p_k| < \frac{M}{k} \qquad (k = N, N+1, \dots). \tag{8.30}$$

On the basis of formulas (8.30) and (8.28), from (8.22) we now derive

$$|b_{2k}| < \frac{M}{k} \qquad (k = N, N+1, \dots). \tag{8.31}$$

---

* For positive constants we shall use in this chapter the same symbol M.

Proceeding again from formulas (8.24) and (8.29), we repeat the previous reasoning, but instead of the boundedness of b_{2k}, we use the stronger statement (8.31). We then obtain

$$|b_{2k}| < \frac{M}{k^2}.$$

Applying now the resultant inequality, we find

$$|b_{2k}| < \frac{M}{k^3}. \tag{8.32}$$

On the basis of (8.32), (8.13) and (8.17), from (8.12) we have

$$|\rho^{-2k} a_{-2k}| < \frac{M}{k^3} \qquad (k = N, N+1, \ldots).$$

Consequently

$$|\rho^{2k} a_{2k}| < \frac{M}{|k|^3} \qquad (k = \pm 1, \pm 2, \ldots).$$

The previous reasoning, as applied to relation (8.23), obviously gives inequalities of the same form for the unknowns b_{2k+1}. Hence, we finally have

$$|\rho^k a_k| < \frac{M}{|k|^3} \qquad (k = \pm 1, \pm 2, \ldots). \tag{8.33}$$

Inequalities (8.33) ensure absolute and uniform convergence of series (8.8), (8.9).

Thus, it is proved that if system (8.14), (8.15) has a bounded solution, the latter gives the solution to our problem. The existence of such a solution for system (8.14), (8.15) can be deduced without difficulty from the general theory of regular systems if (in the case when the system is not completely regular) use is made of the property of a bounded solution proved above and of the uniqueness of the solution to the boundary value problem under consideration.

If we are able to find the solution b_k of the system of equations (8.14), (8.15), then, using formulas (8.13) and (8.12), we can determine the values of all unknown coefficients a_k ($k = 0, \pm 1, \pm 2, \ldots$). Knowing the latter, we can determine, by formulas (8.3), the values of the functions $\varphi_1(t) = \varphi(\sigma)$, $\psi_1(t) = \psi(\sigma)$ on the circumference γ.

After some computation we obtain

$$\varphi(\sigma) = a_0 + \sum_1^\infty a_k (\sigma^k + \sigma^{-k}),$$

$$\psi(\sigma) = \bar{a}_0 - \rho^2 a_2 + \sum_1^\infty [a_{-k}\rho^{-2k} - k\rho^{-2} a_k - (k+2)\rho^2 a_{k+2}] (\sigma^k + \sigma^{-k}).$$

Substituting $t = R\left(\sigma + \frac{1}{\sigma}\right)$, we find $\varphi_1(t)$ and $\psi_1(t)$. When the functions $\varphi_1(z)$ and $\psi_1(z)$ inside the ellipse have been determined, the value of the unknown deflection at the point $z = x + iy$ is found from the familiar formula

$$u(x, y) = w_0(z, \bar{z}) + 2\mathrm{Re}\,\{\bar{z}\varphi_1(z) + \chi_1(x)\}.$$

In particular, the deflection at the middle of the plate is given by

$$u_0 = w_0(0, 0) - [w_0(t, \bar{t}) + 2\mathrm{Re}\,\{\bar{t}\varphi_1(t) + \chi_1(t)\}] \qquad (\chi_1(0) = 0). \tag{8.34}$$

3°. **On the approximate solution of systems of equations.*** We shall neglect, beginning with a sufficiently large N, higher order terms in Eqs. (8.14), (8.15), retaining of all unknowns in each equation only the left-hand side of system (8.18).** In other words, Eqs. (8.14), (8.15), beginning with a certain N, are replaced by the following:

$$\rho^2 b_{k+2} - (\rho^4 + 1)\, b_k + \rho^2 b_{k-2} = B_k \quad (k = 2N, 2N + 1, \ldots). \tag{8.35}$$

We shall consider an infinite system of equations consisting of Eqs. (8.35) and two finite systems each of N first equations of (8.14) and (8.15).

According to formulas (8.22) and (8.23) the bounded solution of system (8.35) is of the form

$$b_{2k} = \frac{1}{1-\rho^4}\left[\sum_{m=k+1}^{\infty} \rho^{2(k-m)} B_{2m} + \sum_{m=N}^{k} \rho^{-2(k-m)} B_{2m}\right] + C_2 \rho^{-2(k+1)}, \tag{8.36}$$

$$b_{2k+1} = \frac{1}{1-\rho^4}\left[\sum_{m=k+1}^{\infty} \rho^{2(k-m)} B_{2m+1} + \sum_{m=N}^{k} \rho^{-2(k-m)} B_{2m+1}\right] + C_2' \rho^{-2(k+1)}, \tag{8.37}$$

where C_2, C_2' are constants, not yet determined.

The constants C_2, C_2' are to be determined together with the first unknowns b_k from the lacking equations of the system under consideration and formulas (8.36), (8.37).

Let us take a system of $N + 1$ first equations of (8.15) and replace the unknowns b_{2N+1}, b_{2N+3} in the last two equations of this finite system by their values from formula (8.37):

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* It is well known that the solution of a regular infinite system is found by a limiting process in the solution of the finite system obtained from the infinite one by rejecting all equations and unknowns beginning with a certain number N. The solution of such a system or, more exactly, any finite number of the first unknowns, can therefore always be found with any prescribed accuracy.

The method proposed below for the solution of the regular system (8.14), (8.15) is more efficient in that, by solving the finite system, we find for a given N approximate values simultaneously for all unknowns b_k $(k = 0, 1, \ldots)$.

** As seen from formula (8.19), the greatest error introduced by this simplification of the system consists in neglecting the quantity $(1 - \rho^4)\dfrac{b_k}{\lambda k}$ as against $(1 + \rho^4)\, b_k$ in the kth equation.

$$b_{2N+1}=\frac{1}{1-\rho^4}\sum_{m=N}^{\infty}\rho^{2(N-m)}B_{2m+1}+C_2'\rho^{-2(N+1)},$$

$$b_{2N+3}=\frac{1}{1-\rho^4}\sum_{m=N+1}^{\infty}\rho^{2(N-m+1)}B_{2m+1}+\frac{\rho^{-2}}{1-\rho^4}B_{2N+1}+C_2'\rho^{-2(N+2)}.$$

For the determination of b_1, b_3, . . ., b_{2N-1}, C_2' we then have a system of $N+1$ equations. By solving this system, we find the first unknowns b_{2k+1} ($k=0, 1, \ldots, N-1$) and also the constant C_2' with the help of which all other b_{2k+1} ($k=N, N+1, \ldots$) are determined from formula (8.37).

By considering a system of $N+1$ first equations of (8.14) and replacing b_{2N}, b_{2N+2} in the last two equations of this system by their values from (8.36), we obtain a complete system for the determination of b_0, b_2, . . ., b_{2N-2}, C_2. By solving this system, we find, in a similar way to that already described, all unknowns b_{2k} ($k=0, 1, \ldots$).

Thus, the simplified system is solved in closed form. It can easily be verified that its solution $b_k^{(N)}$ ($k=0, 1, 2, \ldots$) gives an approximate solution to system (8.14), (8.15).

Example. Elliptical plate under constant load. In the case under consideration the particular solution $w_0(z, z)$ may be taken as

$$w_0(z, \bar{z})=\frac{q}{64D}z^2\bar{z}^2 \qquad (q=\text{constant}).$$

Taking into account a formula resulting from (8.2)

$$f_1(t)=-\lambda\left[\frac{\partial^2}{\partial t\,\partial\bar{t}}+\bar{t}''\frac{\partial}{\partial\bar{t}}\right]w_0(t, \bar{t}) \qquad (\text{on } L),$$

we easily find the following expressions for the coefficients in expansion (8.10):

$$\left.\begin{aligned}
&A_1=[2\lambda(\rho^6+\rho^{-2})-(\rho^4+2)(\rho^2-\rho^{-2})]\frac{qR^3}{32D},\\
&A_3=[2\lambda\rho^{-2}-\rho^2+\rho^{-2}]\frac{qR^3}{32D}, \qquad A_5=-\lambda\rho^{-2}\frac{qR^3}{16D},\\
&A_{-1}=-[\lambda(\rho^6+\rho^{-2})+(\rho^4+1)(\rho^2-\rho^{-2})]\frac{qR^3}{16D},\\
&A_{-3}=-[2\lambda\rho^6+\rho^2(\rho^4-1)]\frac{qR^3}{32D}, \qquad A_{-5}=\lambda\rho^6\frac{qR^3}{16D};
\end{aligned}\right\} \tag{8.38}$$

all other A_k are zero, and in particular $A_{2k}=0$ ($k=1, 2, \ldots$).

Hence, inasmuch as the homogeneous system of equations corresponding to Eqs. (8.14), (8.15) has no non-trivial bounded solutions for $\rho>1$, we have $a_{2k}=0$ ($k=0, 1, 2, \ldots$).

Next, we take $N = 3$ in equalities (8.35). We then have a system of four equations for determining b_1, b_3, b_5, C'_2. Inserting expressions (8.38) in this system and solving it, we find the above-mentioned unknowns, and then by formula (8.37) which has, in our case, the form

$$b_{2k+1} = C'_2\rho^{-2(k+1)} \qquad (k = 3, 4, \ldots)$$

we determine the remaining b_{2k+1}. Thereafter, from formulas (8.13) and (8.12) we determine approximate values of all a_k $(k = \pm 1, \pm 3, \ldots)$.

Rejecting, by way of example, all coefficients a_k $(k = \pm 9, \pm 11, \ldots)$ we have the following approximate expressions* for $\varphi(\sigma)$ and $\psi(\sigma)$:

$$\varphi(\sigma) \cong \sum_{k=1,\,3}^{7} a_k(\sigma^k + \sigma^{-k}),$$

$$\psi(\sigma) \cong \sum_{k=1,\,3,\,5} [a_{-k}\rho^{-2k} - 2\rho^{-2}ka_k - \rho^2(k+2)](\sigma^k + \sigma^{-k}) + \\ + (a_{-7}\rho^{-14} - 7\rho^{-2}a_7)(\sigma^7 + \sigma^{-7}).$$

Substituting the values

$$\sigma + \frac{1}{\sigma} = \frac{t}{R}, \qquad \sigma^3 + \frac{1}{\sigma^3} = \frac{t^3}{R^3} - \frac{3t}{R},$$

$$\sigma^5 + \frac{1}{\sigma^5} = \frac{t^5}{R^5} - \frac{5t^3}{R^3} + \frac{5t}{R},$$

$$\sigma^7 + \frac{1}{\sigma^7} = \frac{t^7}{R^7} - \frac{7t^5}{R^5} + \frac{14t^3}{R^3} - \frac{7t}{R},$$

we find approximate expressions for $\varphi_1(z)$ and $\psi_1(z)$, whereupon the deflections are determined from the above formulas.

The foregoing calculations have been carried out for an ellipse with the ratio of the semi-axes $\frac{a}{b} = 3$. In this case, obviously, $\frac{a}{R} = \frac{3}{\sqrt{2}}$, and consequently, by virtue of formula (8.6), $\rho = \sqrt{2}$. Furthermore, it has been taken that $\nu = \frac{1}{3}$, and hence $\lambda = 3$. By substituting the approximate values of $\varphi_1(z)$, $\chi_1(z)$ thus found in formula (8.34) and setting $t = R(\rho e^{i\pi/3} + \rho^{-1}e^{-i\pi/3})$, the deflection at the middle of the plate is obtained as

$$u_0 \cong 0.2066\frac{qb^4}{D}.$$

Note that this value for u_0 is close to that found by B. G. Galerkin ([25], p. 337).

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* On the basis of formulas (8.38) the coefficients a_k $(k = \pm 1, \pm 3, \ldots)$ are real numbers.

9. Equilibrium of a Confocal Elliptical Ring (Method 2)

The present section is concerned with the mixed-type problem of the bending of a plate in the form of a confocal elliptical ring when one of the contours is fixed and the other is free. As a special case, the first fundamental problem of the plane theory of elasticity is considered for this region. An example is given for the case of a constant load.

1°. **Reduction to an infinite system of equations.** The region of the plate will be denoted as before by S, and its inner and outer boundaries by L_1 and L_2, respectively; the major and minor semi-axes of the corresponding ellipses are a, b and l, d, respectively. We assume that the contour L_2 is fixed and L_1 is free.

For the unknown holomorphic functions $\varphi_1(z)$, $\psi_1(z)$ we have [according to formulas (3.9) and (3.12)] the contour conditions

$$\left.\begin{aligned} &\overline{\varphi_1(t)}+\bar{t}\varphi_1'(t)+\psi_1(t)=f_1(t) \text{ on } L_2,\\ &-\varkappa\overline{\varphi_1(t)}+\bar{t}\varphi_1'(t)+\psi_1(t)=f_1(t)+ic_1t+C \text{ on } L_1,\end{aligned}\right\} \tag{9.1}$$

where $f_1(t)$ is a given function of the arc length s on the complete boundary of the plate; c_1 and C denote, respectively, a real and a complex constant which are to be determined. In the following it will be assumed that $f_1(t)$ has the second derivative with respect to the arc length s which satisfies the Hölder condition of order one.

After transforming the given elliptical ring into a circular one in the ζ plane by the function

$$z=R\left(\zeta+\frac{m}{\zeta}\right),\qquad 2R=\sqrt{\frac{a^2-b^2}{m}}\qquad (0<m<1), \tag{9.2}$$

conditions (9.1) take the form

$$\left.\begin{aligned} &\overline{\varphi(\sigma)}+\left(\frac{\rho^2}{\sigma}+\frac{m\sigma}{\rho^2}\right)\frac{\varphi'(\sigma)}{1-\dfrac{m}{\sigma^2}}+\psi(\sigma)=f(\sigma) \text{ on } \gamma_2,\\ &-\varkappa\overline{\varphi(\sigma)}+\left(\frac{1}{\sigma}+m\sigma\right)\frac{\varphi'(\sigma)}{1-\dfrac{m}{\sigma^2}}+\psi(\sigma)=\\ &\qquad\qquad=f(\sigma)+ic\left(\frac{1}{\sigma}+m\sigma\right)+C \text{ on } \gamma_1;\end{aligned}\right\} \tag{9.3}$$

here γ_1, γ_2 are concentric circles in the ζ plane of radii, respectively, 1 and $\rho>1$,

$$\varphi_1(z)=\varphi(\zeta),\qquad \psi_1(z)=\psi(\zeta),\qquad f_1(t)=f(\sigma),$$
$$c=c_1R,\quad \rho=\frac{l+\sqrt{l^2-4mR^2}}{2R}. \tag{9.4}$$

The functions $\varphi(\zeta)$, $\psi(\zeta)$ will be sought in the form

$$\varphi(\zeta)=\sum_{-\infty}^{\infty}\bar{a}_k\zeta^k, \qquad \left(1-\frac{m}{\zeta^2}\right)\psi(\zeta)=\sum_{-\infty}^{\infty}a'_k\zeta^k. \tag{9.5}$$

Hence ($\zeta=re^{i\vartheta}$),

$$\overline{\varphi(\zeta)}=\sum_{-\infty}^{\infty}\bar{a}_kr^ke^{-ik\vartheta}, \qquad \varphi'(\zeta)=\sum_{-\infty}^{\infty}ka_kr^{k-1}e^{(k-1)i\vartheta}. \tag{9.6}$$

Assume also

$$\left(1-\frac{m}{\sigma^2}\right)f(\sigma)=\begin{cases}\sum\limits_{-\infty}^{\infty}A'_ke^{ik\vartheta} & \text{on } \gamma_1,\\ \sum\limits_{-\infty}^{\infty}A''_ke^{ik\vartheta} & \text{on } \gamma_2.\end{cases} \tag{9.7}$$

Inserting the above series in expressions (9.3), comparing the coefficients of $e^{ik\vartheta}$ ($k=0, \pm 1, \ldots$) and eliminating a'_k from the resulting equalities, we obtain the following system of equations for determining the unknowns c, C, a_k:

$$m(\varkappa+\rho^{-4})\bar{a}_{-2}+2(1-\rho^2)a_2=B_0+C, \tag{9.8}$$

$$-(\varkappa+\rho^4)\bar{a}_2+2m(\rho^{-2}-1)a_{-2}=B_{-2}-mC, \tag{9.9}$$

$$m(\varkappa+\rho^{-2})\bar{a}_{-1}+(1-\rho^2)a_1-(\varkappa+\rho^2)\bar{a}_1-m(1-\rho^{-2})a_{-1}=$$
$$=B_{-1}+i(1-m^2)c, \tag{9.10}$$

$$m(\varkappa+\rho^{-6})\bar{a}_{-3}+3(1-\rho^2)a_3-(\varkappa+\rho^{-2})\bar{a}_{-1}+m(1-\rho^{-2})a_1=$$
$$=B_1+imc, \tag{9.11}$$

$$-(\varkappa+\rho^6)\bar{a}_3+3m(\rho^{-2}-1)a_{-3}+m(\varkappa+\rho^2)\bar{a}_1-(1-\rho^2)a_{-1}=$$
$$=B_{-3}-imc, \tag{9.12}$$

$$m(\varkappa+\rho^{-2k})\bar{a}_{-k}+k(1-\rho^2)a_k-(\varkappa+\rho^{-2k+4})\bar{a}_{-k+2}+$$
$$+m(k-2)(1-\rho^{-2})a_{k-2}=B_{k-2}, \tag{9.13}$$

$$-(\varkappa+\rho^{2k})\bar{a}_k-mk(1-\rho^{-2})a_{-k}+m(\varkappa+\rho^{2k-4})\bar{a}_{k-2}-$$
$$-(k-2)(1-\rho^2)a_{-k+2}=B_{-k} \qquad (k=4, 5, \ldots), \tag{9.14}$$

$$B_k=A'_k-\rho^{-k}A''_k \qquad (k=0, \pm 1, \pm 2, \ldots). \tag{9.15}$$

The recurrence relations (9.8) to (9.14) enable one to determine successively the unknowns a_{2k}, a_{-2k} in terms of B_{2s-2}, B_{-2s} and C, and the unknowns a_{2k-1}, a_{-2k+1} in terms of B_{2s-3}, B_{-2s+1}, c and a_1 ($s=1, 2, \ldots, k$). Hence, in view of the fact that the constants C, c, a_1 (just as all other a_k) are unknowns, the system of equations (9.8) to (9.14) admits infinitely many solutions. Of this set of solutions we have to choose, of course, only those for which series (9.6) will be convergent on γ_1 and γ_2.

We seek a solution to the infinite system (9.8) to (9.14) in a class of bounded sequences of numbers and set ourselves the aim to prove that this class of solutions does lead to the aim. We first reduce our system to a somewhat different form.

Assume

$$(\varkappa+\rho^{-2k})\,\bar{a}_{-k}+k\,\frac{1-\rho^2}{m}\,a_k=\Omega_k \qquad (k=1,\ 2,\ \ldots). \qquad (9.16)$$

Then the set of equalities (9.11) and (9.13) with odd numbers is rewritten as

$$\left.\begin{aligned} &m\Omega_3-\Omega_1=B_1+imc-\frac{\lambda}{m}\,a_1,\\ &m\Omega_{2k-1}-\Omega_{2k-3}=B_{2k-3}-(2k-3)\,\frac{\lambda}{m}\,a_{2k-3} \quad (k=3,\ 4,\ \ldots),\end{aligned}\right\} \qquad (9.17)$$

where

$$\lambda=(1-\rho^2)\left(1-\frac{m}{\rho^2}\right).$$

By solving the above system for Ω_{2k-1}, we have

$$\Omega_{2k-1}=[\Omega_1+imc]\,m^{-k+1}-\lambda\sum_{s=1}^{k-1} m^{-k+s-1}\,(2s-1)\,a_{2s-1}+$$
$$+\sum_{s=1}^{k-1} m^{-k+s}B_{2s-1} \qquad (k=2,\ 3,\ \ldots). \qquad (9.18)$$

We introduce into consideration the following Fourier coefficients:

$$A_k=\frac{1}{2\pi i}\int\limits_{\gamma}\frac{f(\sigma)\,d\sigma}{\sigma^{k+1}}, \quad \gamma=\gamma_1+\gamma_2 \qquad (k=0,\ \pm 1,\ \pm 2,\ \ldots). \qquad (9.19)$$

Then, as is easily seen from relation (9.7),

$$mA_k-A_{k-2}=B_{k-2} \qquad (k=0,\ \pm 1,\ \ldots). \qquad (9.20)$$

From this we find

$$A_{2k-1}=A_1m^{-k+1}+\sum_{s=1}^{k-1} m^{-k+s}B_{2s-1} \qquad (k=1,\ 2,\ \ldots).$$

Inserting this expression in relation (9.18), we obtain

$$\Omega_{2k-1}+m^{-k}\left[\lambda\sum_{s=1}^{k-1} m^{s-1}\,(2s-1)\,a_{2s-1}-m\,(\Omega_1+imc-A_1)\right]=$$
$$=A_{2k-1} \qquad (k=2,\ 3,\ \ldots). \qquad (9.21)$$

On the basis of formula (9.16), for a bounded solution of system (9.8) to (9.14) for large k we have: $\Omega_k = O(k)$. Hence, by virtue of relation (9.21) we must have (remembering that $0 < m < 1$)

$$m[\Omega_1 + imc - A_1] = \lambda \sum_{s=1}^{\infty} m^{s-1}(2s-1)a_{2s-1}. \tag{9.22}$$

After this equalities (9.21) become

$$\Omega_{2k-1} + \lambda m^{-k}\left[\sum_{s=1}^{k-1} m^{s-1}(2s-1)a_{2s-1} - A\right] = \\ = A_{2k-1} \qquad (k = 2, 3, \ldots), \tag{9.23}$$

$$A = \sum_{s=1}^{\infty} m^{s-1}(2s-1)a_{2s-1}. \tag{9.24}$$

Assume further

$$(\varkappa + \rho^{2k})\bar{a}_k + mk(1-\rho^{-2})a_{-k} = \Omega_k^* \qquad (k = 1, 2, \ldots) \tag{9.25}$$

and rewrite equalities (9.14) as

$$\Omega_k^* - m\Omega_{k-2}^* = -B_{-k} - (k-2)\lambda a_{-k+2} \qquad (k = 4, 5, \ldots). \tag{9.26}$$

Consider the system of equations (9.10), (9.12) and (9.26) with odd numbers, namely

$$\left.\begin{aligned} &\Omega_1^* - m\Omega_1 = -B_{-1} + i(m^2-1)c, \\ &\Omega_3^* - m\Omega_1^* = -B_{-3} + imc - \lambda a_{-1}, \\ &\Omega_{2k-1}^* - m\Omega_{2k-3}^* = -B_{-2k+1} - (2k-3)\lambda a_{-2k+3} \\ &\qquad\qquad\qquad\qquad\qquad (k = 3, 4, \ldots). \end{aligned}\right\} \tag{9.27}$$

By solving this system for Ω_{2k-1}^*, we have

$$\Omega_{2k-1}^* = m^k\left[\Omega_1 + imc - \lambda\sum_{s=1}^{k-1} m^{-(s+1)}(2s-1)a_{-2s+1} - \sum_{s=1}^{k} m^{-s}B_{-2s+1}\right] \\ (k = 2, 3, \ldots). \tag{9.28}$$

On the other hand, from relation (9.20) we easily obtain

$$A_{-2k+1} = m^k A_1 - \sum_{s=1}^{k} m^{k-s}B_{-2s+1} \qquad (k = 1, 2, \ldots).$$

Inserting this expression in equality (9.28) and replacing Ω_1 by its value from relation (9.22), there results

$$\Omega_{2k-1}^* + \lambda m^{k-1}\left[\sum_{s=1}^{k-1} m^{-s}(2s-1)a_{-2s+1} - A\right] = A_{-2k+1} \\ (k = 2, 3, \ldots). \tag{9.29}$$

Substituting Ω_1 from relation (9.22) in the first equality of (9.27), we further obtain

$$\Omega_1^* - \lambda A = A_{-1} - ic. \tag{9.30}$$

Equalities (9.22), (9.23), (9.29), (9.30) give the required infinite system of linear algebraic equations for determining c and a_{2k-1} ($k = 0, \pm 1, \ldots$). According to formulas (9.16) and (9.25) it is written as follows:

$$\left.\begin{aligned}
&(\varkappa+\rho^2)\bar{a}_1+m(1-\rho^{-2})a_{-1}-\lambda A=A_{-1}-ic,\\
&m(\varkappa+\rho^{-2})\bar{a}_{-1}+(1-\rho^2)a_1-\lambda A=m(A_1-imc),\\
&(\varkappa+\rho^{4k-2})\bar{a}_{2k-1}+(2k-1)m(1-\rho^{-2})a_{-2k+1}+\\
&\qquad+\lambda m^{k-1}\Big[\sum_{s=1}^{k-1}m^{-s}(2s-1)a_{-2s+1}-A\Big]=A_{-2k+1}\\
&\qquad\qquad\qquad\qquad\qquad\qquad (k=2,\ 3,\ \ldots),\\
&(\varkappa+\rho^{-4k+2})\bar{a}_{-2k+1}+(2k-1)\frac{1-\rho^2}{m}a_{2k-1}+\\
&\qquad+\lambda m^{-k}\Big[\sum_{s=1}^{k-1}m^{s-1}(2s-1)a_{2s-1}-A\Big]=A_{2k-1};
\end{aligned}\right\}\qquad(9.31)$$

here

$$A=\sum_{s=1}^{\infty}m^{s-1}(2s-1)a_{2s-1}.$$

By considering equalities (9.13) and (9.14) with even numbers together with equalities (9.8) and (9.9), respectively, we obtain, by an exactly similar reasoning, the following system in C and a_{2k} ($k = \pm 1, \pm 2, \ldots$):

$$\left.\begin{aligned}
&C-\lambda B=A_0,\\
&(\varkappa+\rho^{4k})\bar{a}_{2k}+2mk(1-\rho^{-2})a_{-2k}+\lambda m^k\times\\
&\qquad\times\Big[2\sum_{s=1}^{k-1}m^{-(s+1)}sa_{-2s}-B\Big]=A_{-2k},\\
&(\varkappa+\rho^{-4k})\bar{a}_{-2k}+2k\frac{1-\rho^2}{m}a_{2k}+\\
&\qquad+\lambda m^{-k}\Big[2\sum_{s=1}^{k-1}m^{s-1}sa_{2s}-B\Big]=A_{2k}\qquad(k=1,\ 2,\ \ldots),
\end{aligned}\right\}\qquad(9.32)$$

where

$$B=2\sum_{s=1}^{\infty}m^{s-1}sa_{2s}.$$

As is seen from the very derivation of this system, it is equivalent (in the sense of solvability in the class of bounded solutions) to the original one. The newly obtained system is by no means simpler in structure than the old one, but nevertheless it is found more convenient for investigation.*

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* For the practical solution of the system, however, there is no need, of course, for such reduction.

2°. Investigation of the systems obtained. It will be proved later that system (9.31), (9.32) has one and only one solution c, C, a_k $(k=\pm 1, \pm 2, \ldots)$ such that it gives absolutely and uniformly convergent series for $\varphi(\zeta)$, $\varphi'(\zeta)$, and consequently it will be a solution to the boundary value problem under consideration.* A method for solving this system will also be indicated.

By eliminating the unknowns a_{-2k+1} from system (9.31), we obtain a system in c and a_{2k-1} $(k=1, 2, \ldots)$ of the form

$$\left[\varkappa+\rho^2+\frac{(1-\rho^2)^2}{1+\varkappa\rho^2}\right]a_1-\frac{\lambda(1-\rho^2)}{1+\varkappa\rho^2}A-\lambda\bar{A}+ i\left[\frac{m^2(1-\rho^2)^2}{1+\varkappa\rho^2}-1\right]c=A^*_{-1}, \qquad (9.33')$$

$$\left[\varkappa+\rho^{4k-2}+\frac{(2k-1)^2\alpha\beta}{\varkappa+\rho^{-4k+2}}\right]a_{2k-1}+\frac{(2k-1)\alpha\lambda}{\varkappa+\rho^{-4k+2}}m^{-k}\times$$

$$\times\left[\sum_{s=1}^{k-1}m^{s-1}(2s-1)a_{2s-1}-A\right]-\lambda\beta\sum_{s=1}^{k-1}\frac{m^{k-s-1}(2s-1)^2}{\varkappa+\rho^{-4s+2}}a_{2s-1}-$$

$$-\lambda^2\sum_{s=1}^{k-1}\frac{m^{k-2s-1}(2s-1)}{\varkappa+\rho^{-4s+2}}\left[\sum_{\nu=1}^{s-1}m^{\nu-1}(2\nu-1)a_{2\nu-1}-A\right]-$$

$$-\lambda m^{k-1}\left[\bar{A}+\frac{ic}{\varkappa+\rho^{-2}}\right]=A^*_{-2k+1} \qquad (k=2, 3, \ldots), \qquad (9.33'')$$

where

$$\alpha=-m(1-\rho^{-2}), \quad \beta=\frac{1-\rho^2}{m},$$

$$A^*_{-2k+1}=\bar{A}_{-2k+1}+\frac{(2k-1)\alpha}{\varkappa+\rho^{-4k+2}}A_{2k-1}-$$

$$-\lambda\sum_{s=1}^{k-1}\frac{m^{k-s-1}(2s-1)}{\varkappa+\rho^{-4s+2}}A_{2s-1} \qquad (k=1, 2, \ldots). \qquad (9.34)$$

Owing to the fact that the coefficient of a_{2k-1} in the equalities of the above system considerably outweighs the others for large k, it is possible to establish the order of decreasing of the unknowns in this system as $k\to\infty$.

---

* After the absolute and uniform convergence of the series for $\varphi(\zeta)$, $\varphi'(\zeta)$ is proved, a similar statement in regard to the series for $(1-m\zeta^{-2})\psi(\zeta)$ follows immediately from the boundary conditions (9.3).

Let c, a_{2k-1} ($k = 1, 2, \ldots$) be some (bounded) solution of system (9.33). Assume

$$\left.\begin{aligned} p_k &= \sum_{s=1}^{k-1} \frac{m^{k-s-1}(2s-1)^2}{\varkappa+\rho^{-4s+2}} \,|a_{2s-1}|, \\ q_k &= (2k-1)\, m^{-k} \sum_{s=k}^{\infty} m^{s-1}(2s-1)\,|a_{2s-1}|, \\ r_k &= \sum_{s=1}^{k-1} \frac{m^{k-2s-1}(2s-1)}{\varkappa+\rho^{-4s+2}} \sum_{\nu=s}^{\infty} m^{\nu-1}(2\nu-1)\,|a_{2\nu-1}|, \end{aligned}\right\} \qquad (9.35)$$

$$P_k = \sum_{s=1}^{k-1} \frac{m^{k-s-1}(2s-1)}{\varkappa+\rho^{-4s+2}} \,|A_{2s-1}|. \qquad (9.36)$$

It is easy to see that

$$p_k < Mk^2, \qquad q_k < Mk^2, \qquad r_k < Mk^2 \qquad (M = \text{constant}). \qquad (9.37)$$

Note now that, according to our assumption regarding $f_1(t)$, the following inequalities are valid for the Fourier coefficients (9.19):

$$|A_k| < \frac{M}{k^3}, \quad |\rho^{-k} A_{-k}| < \frac{M}{k^3} \qquad (k = 1, 2, \ldots). \qquad (9.38)$$

On this basis

$$P_k < \frac{M}{(k-1)^2}\left[1 + \left(\frac{k-1}{k-2}\right)^2 m + \left(\frac{k-1}{k-3}\right)^2 m^2 + \ldots + (k-1)^2 m^{k-2}\right],$$

and hence (see Sec. 8, Par. 2°)

$$P_k < \frac{M}{k^2} \qquad (k = 1, 2, \ldots). \qquad (9.39)$$

By using estimates (9.37) to (9.39) and also considering equality (9.24), from (9.33) we find

$$|\rho^{2k-1} a_{2k-1}| < \frac{M}{k^3} \qquad (k = 1, 2, \ldots).$$

With the use of the above inequality and inequalities (9.38), from the last equality of (9.31) we also find

$$|a_{-2k+1}| < \frac{M}{k^3} \qquad (k = 1, 2, \ldots).$$

A similar argument applied to system (9.32) obviously leads to the same inequalities for its bounded solution a_{2k} ($k = \pm 1, \pm 2, \ldots$).

Consequently, if c, a_k ($k = \pm 1, \pm 2, \ldots$) is a bounded solution* of system (9.31), (9.32), then

$$|\rho^k a_k| < \frac{M}{k^3}, \quad |a_{-k}| < \frac{M}{k^3} \qquad (k = 1, 2, \ldots). \qquad (9.40)$$

---

* The boundedness of the sequence a_{-k} ($k = 1, 2, \ldots$) may not necessarily be assumed.

Since series (9.6), subject to condition (9.40), converge uniformly and absolutely for $1 \leqslant |\zeta| \leqslant \rho$, the statement made at the beginning of this paragraph reduces to proving that the system of equations (9.33) [and the analogous system obtained from (9.32)] has a bounded solution. The truth of the latter statement follows immediately from the theory of regular systems if use is made of the above-proved property of the solutions of system (9.31), (9.32) and of the theorem of uniqueness of solution of the boundary value problem under consideration.*

From this we conclude that the system of equations (9.8) to (9.14), subject to condition (9.38), admits a (unique) solution possessing property (9.40), and consequently our boundary value problem is solvable.

3°. **Approximate solution of infinite systems.** We now proceed to the question of the actual solution of system (9.33). Introduce new unknowns b_k by the relation

$$ka_k = b_k \qquad (k = 1, 2, \dots) \tag{9.41}$$

and rewrite this system as

$$\left[\varkappa + \rho^2 + \frac{(1-\rho^2)^2}{1+\varkappa\rho^2}\right] b_1 = \frac{\lambda(1-\rho^2)}{1+\varkappa\rho^2} A + \lambda\bar{A} + \\ + i\left[1 - \frac{m^2(1-\rho^2)}{1+\varkappa\rho^2}\right] c + A^*_{-1}, \tag{9.42′}$$

$$\lambda^*\left[\frac{\varkappa+\rho^{4k-2}}{2k-1} + \frac{(2k-1)(1-\rho^2)^2}{\rho^2(\varkappa+\rho^{-4k+2})}\right] b_{2k-1} + \\ + \left[\frac{2k-1}{\rho^2(\varkappa+\rho^{-4k+2})} - \frac{2k-3}{\varkappa+\rho^{-4k+6}}\right] b_{2k-3} + \\ + \sum_{s=1}^{k-2} m^{k-s-1}\left[\frac{(2k-1)\, m^{-2(k-s-1)}}{\rho^2(\varkappa+\rho^{-4k+2})} - \right. \\ \left. - \frac{2s-1}{\varkappa+\rho^{-4s+2}} - \left(1 - \frac{m^2}{\rho^2}\right)\sum_{\nu=s}^{k-2} \frac{m^{-2(\nu-s+1)}(2\nu+1)}{\varkappa+\rho^{-2(2\nu+1)}}\right] b_{2s-1} = \\ = \left[\frac{(2k-1)\, m^{-k+2}}{\rho^2(\varkappa+\rho^{-4k+2})} - \left(1 - \frac{m^2}{\rho^2}\right)\sum_{s=1}^{k-1} \frac{m^{k-2s}(2s-1)}{\varkappa+\rho^{-4s+2}}\right] A + \\ + \frac{m^k}{1-\rho^2}\left[\bar{A} + \frac{ic}{\varkappa+\rho^{-2}}\right] + \lambda^* A^*_{-2k+1}, \tag{9.42″}$$

---

* It can easily be verified that the system of equations (9.33) (and the system analogous to it) is quasi-regular. Indeed, for this it is sufficient to note that inequalities (9.37) remain true if we set $a_{2s-1} = 1$ $(s = 1, 2, \dots)$ on the right-hand sides of formulas (9.35).

where

$$\lambda^* = \frac{m\rho^2}{(1-\rho^2)^2(\rho^2-m^2)} \quad (k=2,\ 3,\ \ldots), \quad A = \sum_{s=1}^{\infty} m^{s-1} b_{2s-1}. \qquad (9.43)$$

If the constants c and A were known in advance, the above system, being triangular in b_{2k-1} ($k = 1, 2, \ldots$), would enable one to determine successively all b_{2k-1} since the coefficients of the latter are different from zero for every k. In the case considered, however, the solution can be obtained as follows.

Neglecting, for the present, relation (9.43), we shall solve system (9.42) with the right-hand sides, respectively,

$$A = \bar{A} = c = 0; \ \bar{A} = c = A^*_{-2k+1} = 0,\ A = 1; \ A = c = A^*_{-2k+1} = 0,$$
$$\bar{A} = 1 \qquad (k = 1,\ 2,\ \ldots).$$

Let these solutions be, respectively, $b^{(0)}_{2k-1}, b^{(1)}_{2k-1}, b^{(2)}_{2k-1}$. It is obvious that $b^{(s)}_{2k-1}$ ($s = 1, 2$) are real numbers.

Then the solution b_{2k-1} of system (9.42) is represented as

$$b_{2k-1} = b^{(0)}_{2k-1} + Ab^{(1)}_{2k-1} + \left(\bar{A} + \frac{ic}{\varkappa+\rho^{-2}}\right) b^{(2)}_{2k-1} \qquad (k = 1,\ 2,\ \ldots). \qquad (9.44)$$

Multiplying these equalities by m^{k-1}, summing over k from 1 to ∞* and taking into account formula (9.43), we have

$$A = \sum_{k=1}^{\infty} m^{k-1} b^{(0)}_{2k-1} + A \sum_{k=1}^{\infty} m^{k-1} b^{(1)}_{2k-1} +$$
$$+ \left(\bar{A} + \frac{ic}{\varkappa+\rho^{-2}}\right) \sum_{k=1}^{\infty} m^{k-1} b^{(2)}_{2k-1}. \qquad (9.45)$$

In view of the fact that the unknown b_1 may be considered as a real quantity, without loss of generality, the above equality, together with equality (9.44) for $k = 1$ or, which amounts to the same thing, with the first equality of (9.42), represents a complete system for determining b_1, c, A. This finite system is uniquely solvable by reason of the uniqueness of solution of our boundary value problem.

After determining b, c, A in this manner, we find the unknowns b_{2k-1}, beginning with $k = 2$, from formula (9.44), whereupon a_{-2k+1} are obtained from the second equalities of (9.31).

In an exactly similar way we can solve the system for b_{2k} ($k = 1, 2, \ldots$) containing, besides the known quantities, the unknowns

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* It is easily verified that the corresponding series will be absolutely convergent.

B and $\overline{B}$ on the right-hand side; for determining the latter we have one (complex) equality analogous to (9.45).*

When the solution of systems (9.31), (9.32) has been found, the coefficients a_k' in the expansion of the function $(1 - m\zeta^{-2})\,\psi(\zeta)$ are determined directly from the boundary conditions (9.3). Indeed, substitute series (9.5) to (9.7) in the first condition of (9.3) and compare the coefficients of $e^{ik\vartheta}$ $(k = 0, \pm 1, \ldots)$. We have

$$\begin{gathered}\overline{a}_{-k}\rho^{-2k} - m\overline{a}_{-k-2}\rho^{-2(k+2)} + (k+2)\,\rho^2 a_{k+2} + \\ + \frac{m}{\rho^2}\,ka_k + a_k' = \rho^{-k}A_k'' \qquad (k = 0, \pm 1, \pm 2, \ldots). \qquad (9.46)\end{gathered}$$

These equalities uniquely determine all a_k' if a_0 is fixed.

As seen from the above, all unknowns of the problem under consideration are determined by solving two triangular systems of the form (9.42).

We return to the question of finding the solution of our infinite systems. It will be seen later that the foregoing method of constructing the solution of system (9.42) admits a considerable simplification in the practical solution.

We write this system in the form

$$\begin{gathered}\left[\frac{\varkappa + \rho^{4k-2}}{2k-1} + \frac{(2k-1)(1-\rho^2)^2}{\rho^2(\varkappa + \rho^{-4k+2})}\right] b_{2k-1} + \\ + F_{2k-1}(b_1, b_3, \ldots, b_{2k-3}, c, A) = A^*_{-2k+1} \\ (k = 1, 2, \ldots). \qquad (9.47)\end{gathered}$$

On the basis of the first inequality of (9.40) it is easy to establish the following estimates [see formulas (9.35)]:

$$p_k, r_k < \begin{cases} Mm^k & \text{when } m\rho^2 > 1, \\ Mk^{-1}\rho^{-2k} & \text{when } m\rho^2 < 1, \\ M\rho^{-2k}\ln k & \text{when } m\rho^2 = 1. \end{cases}$$

By virtue of these inequalities

$$\begin{gathered}|F_{2k-1}(b_1, b_3, \ldots, b_{2k-3}, c, A)| < Mm_0^k \ln k, \qquad (9.48) \\ m_0 = \max\left\{m, \frac{1}{\rho^2}\right\} \qquad (k = 2, 3, \ldots).\end{gathered}$$

On the other hand, inequalities (9.40), (9.38) and (9.39) give

$$\left.\begin{aligned}&\left|\left[\frac{\varkappa + \rho^{4k-2}}{2k-1} + \frac{(2k-1)(1-\rho^2)^2}{\rho^2(\varkappa + \rho^{-4k+2})}\right] b_{2k-1}\right| < \frac{M}{k^3}\rho^{2k}, \\ &|A^*_{-2k+1}| < \frac{M}{k^3}\rho^{2k} \qquad (k = 2, 3, \ldots).\end{aligned}\right\} \qquad (9.49)$$

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* For details the reader is referred to the author's paper [39].

The foregoing estimates enable us to replace equalities (9.47), beginning with a sufficiently large N, by the following:*

$$\left[\frac{\varkappa+\rho^{4k-2}}{2k-1}+\frac{(2k-1)(1-\rho^2)^2}{\rho^2(\varkappa+\rho^{-4k+2})}\right]b_{2k-1}=A^*_{-2k+1}$$

$$(k=N,\ N+1,\ \ldots). \qquad (9.50)$$

This simplified system consisting of the first $N-1$ equations of (9.42) and equalities (9.50) admits a solution in finite form. Here one has, in fact, to solve only a finite system of triangular form. Equalities (9.44), beginning with $k=N$, will be identical with system (9.50). It is obvious that, by increasing N, we can find in this way the solution of system (9.42) with any accuracy. All that was said about system (9.42) applies equally well to the system for b_{2k}, and thereby the solution of the problem is completed.

4°. **Plane problem for an elliptical ring.** On putting $\varkappa=-1$, $c=0$ in formulas (9.1) we obtain the first fundamental problem of the plane theory of elasticity for the region under consideration.** Of course, the systems of linear algebraic equations for determining the coefficients in expansion (9.6) are obtained here from the corresponding systems of the case considered by substituting $\varkappa=-1$, $c=0$.

The first equation of (9.42) becomes

$$-\lambda\,[A+\bar{A}]=\bar{A}_{-1}+mA_1. \qquad (9.51)$$

Equation (9.51) entails the necessity of the condition

$$\operatorname{Im}\{\bar{A}_{-1}+mA_1\}=0$$

which, as is easily seen, means that the resultant couple of all external forces is zero.

In the case considered, equalities (9.44) take the form

$$b_{2k-1}=b^{(0)}_{2k-1}+Ab^{(1)}_{2k-1}+\bar{A}b^{(2)}_{2k-1} \qquad (k=2,\ 3,\ \ldots),$$

---

* Such a simplification of our system means that we neglect quantities of order $km_0^k\rho^{-4k}\ln k$ in the expressions for the unknowns b_{2k-1}, beginning with a certain number.

It will be recalled that with inequalities (9.38) the solution b_{2k-1} of system (9.42) is a small quantity of order not higher than $k^{-2}\rho^{-2k}$ with increasing k. Of course, the same applies to the solution of system (9.50).

** Attempts to solve this problem remained futile for a long time. A method for its approximate solution, well suited for practical purposes, was first stated in the work of M. P. Sheremet'ev [115] published in 1953. By combining the method of functional equations with the power series method, the aforenamed author reduced the problem to infinite systems of linear equations, somewhat more complicated outwardly but similar in structure to those proposed above, indicated a procedure for their approximate solution and carried the solution to numerical results in one specific case. The question of the justification of the algorithm was not examined by M. P. Sheremet'ev.

where, as before, $b_{2k-1}^{(1)}$, $b_{2k-1}^{(2)}$ are real numbers. By multiplying these equalities by m^{k-1} and summing, we obtain, by formula (9.43),

$$A-b_1=\sum_{k=2}^{\infty} m^{k-1}b_{2k-1}^{(0)}+A\sum_{k=2}^{\infty} m^{k-1}b_{2k-1}^{(1)}+\bar{A}\sum_{k=2}^{\infty} m^{k-1}b_{2k-1}^{(2)}. \quad (9.52)$$

In view of the fact that b_1 is a real quantity, equalities (9.51), (9.52) uniquely determine b_1 and A. Thereafter a_{-2k+1} $(k=1, 2, \ldots)$ are determined from the corresponding equalities of system (9.31), the first two equalities of which are identical by virtue of Eq. (9.51).

We have next to show that the first coefficients in equalities (9.42) and in the analogous equalities for b_{2k}, namely the expressions of the form (apart from a constant factor)

$$\frac{1-\rho^{2k}}{k}+\frac{k(1-\rho^2)^2}{\rho^2(1-\rho^{2k})} \qquad (k=2, 3, \ldots)$$

are different from zero for $\rho>1$. This is proved in perfect analogy to the corresponding statement given by N. I. Muskhelishvili in [93], p. 210. All the rest remains unaltered.

10. Example. Elliptical Ring under Constant Pressure

Consider in greater detail the case when the load $q(x, y)$ is constant. The contour conditions (9.1) in this case may be represented as

$$\left.\begin{aligned} &\overline{\varphi_0(t)}+\bar{t}\varphi_0'(t)+\psi_0(t)=f_0(t) \quad \text{on} \quad L_2, \\ &-\varkappa\overline{\varphi_0(t)}+\bar{t}\varphi_0'(t)+\psi_0(t)=f_0(t)+ic_1(t)+C \quad \text{on} \quad L_1, \end{aligned}\right\} \quad (10.1)$$

where $\varphi_0(z)$, $\psi_0(z)$ are the unknown analytic functions in the given ring,

$$\left.\begin{aligned} &f_0(t)=-\frac{\partial w_0}{\partial t} \quad \text{on} \quad L_2, \\ &f_0(t)=\frac{1}{2(1-\nu)}\int^{\bar{t}}\left[G(w_0)-i\int^{s}H(w_0)\,ds_1\right]d\bar{t} \quad \text{on} \quad L_1, \end{aligned}\right\} \quad (10.2)$$

G and H are the differential operators (1.23),

$$w_0(z, \bar{z})=\frac{q}{64D}z^2\bar{z}^2.$$

Considering the nature of the multiple-valuedness of the functions $\varphi_0(z)$, $\psi_0(z)$ (S. G. Lekhnitskii [73], p. 209), we find from the second equalities of (10.1) and (10.2) after elementary computations

$$\varphi_0(z) = -\frac{1}{16D}\,\frac{1+m}{1-m}\,qb^2\Big[\ln(1-m)+\ln\frac{z}{b}\Big]z+\varphi_1(z),$$

$$\psi_0(z) = \psi_1(z), \tag{10.3}$$

where $\varphi_1(z)$, $\psi_1(z)$ are holomorphic functions in the given region.

Inserting expressions (10.3) in formula (10.1) and substituting for t from formula (9.2), we obtain the following expressions for the right-hand sides of conditions (9.3):

$$f(\sigma)=\frac{1}{32(1-m)^2}\Big\{2(1+m)\Big[\Big(\frac{m\sigma}{\rho^2}+\frac{\rho^2}{\sigma}\Big)(1+2\ln\rho)+$$

$$+\ln\Big(1+\frac{m}{\sigma^2}\Big)\Big(1+\frac{m\sigma^2}{\rho^4}\Big)\Big]-$$

$$-\frac{1}{1-m}\Big(\sigma+\frac{m}{\sigma}\Big)\Big(\frac{m\sigma}{\rho^2}+\frac{\rho^2}{\sigma}\Big)^2\Big\}\frac{qb^3}{D} \quad \text{on} \quad \gamma_2, \tag{10.4'}$$

$$f(\sigma)=\frac{1}{16(1-m)^2}\Big\{(1+m)\Big(m\sigma+\frac{1}{\sigma}\Big)\Big[1+\ln\Big(1+\frac{m}{\sigma^2}\Big)-$$

$$-\varkappa\ln(1+m\sigma^2)\Big]+\frac{1}{1-m}\Big[\frac{1+\nu}{1-\nu}\Big(\frac{m^2\sigma^3}{3}+m^3\sigma+\frac{1}{\sigma}+\frac{m}{3\sigma^3}\Big)-$$

$$-\frac{1}{2}\Big(\frac{m^2\sigma^3}{3}+[(2m-m^3)\sigma+\frac{2m^2-1}{\sigma}+\frac{m}{3\sigma^3}\Big)+$$

$$+\frac{4(1-m^2)}{1-\nu}\Big(m\sigma-\frac{1}{\sigma}\Big)\Big]\Big]\Big\}\frac{qb^3}{D} \quad \text{on} \quad \gamma_1. \tag{10.4''}$$

By expanding this function in powers of σ, we obtain the coefficients A_k ($k = 0, \pm 1, \ldots$) by formula (9.19).

By the argument used previously the unknown constants c, a_k are to be determined from system (9.42) and the analogous system, whereupon all other unknowns are found. As is seen from formula (10.4), in our case $A_{2k} = A'_{2k} = A''_{2k} = 0$ ($k = 0, \pm 1, \ldots$), and hence $C = 0$, $a_{2k} = a'_{2k} = 0$ ($k = 0, \pm 1, \ldots$). Moreover, since all A_k are real, $c = 0$, $A = \bar{A}$ and system (9.42) is simplified.

In this way the calculation was carried out for a ring $\frac{a}{b} = \frac{d}{b} = 2$, with the first four equations retained in system (9.42); the coefficients a_k, a'_k ($k = \pm 8, \pm 9, \ldots$) were rejected. In the example considered here, $m = \frac{1}{3}$, $\rho^2 = (11 + 4\sqrt{7})/9$. Furthermore, it was taken that $\nu = \frac{1}{3}$.

As a result of these calculations, for the bending moment G_0 along the contour of the plate ($G_0 = -DG$) expressed in terms of the functions $\varphi_0(z)$, $\psi_0(z)$ as

$$G_0=\operatorname{Re}\Big\{2(1-\nu)\frac{d}{dt}\,[-\varkappa\varphi_0(t)+t\overline{\varphi_0'(t)}+\overline{\psi_0(t)}]-$$

$$-\frac{q}{8}\Big[(1+\nu)\,t\bar{t}-\frac{1-\nu}{2}\,\bar{t}^2\Big(\frac{dt}{ds}\Big)^2\Big]\Big\}$$

the following values (in qb^2) are obtained at the vertices L_1 and L_2:

$G_0 = 0.0066$ for $t = a$, $\quad G_0 = 0.0030$ for $t = ib$,

$G_0 = -0.2000$ for $t = l$, $\quad G_0 = -0.3681$ for $t = id$.

CHAPTER THREE

A Direct Method for the Solution of One Class of Singular Equations and Its Applications

The problem of contact between two bodies (one of which may be absolutely rigid) admits a solution in closed form if the region of contact between them is small compared with the dimensions of the bodies or when their boundaries are close to straight lines. In these cases the problem can be reduced in one way or another to the so-called characteristic singular equation whose solution is found in explicit form. Among these problems are the fundamental mixed problem for a half-plane, the problem of the pressure of a rigid punch on the boundary of a half-plane with, and without, friction, the generalized Hertz plane problem of pressure between two elastic bodies, and many others. It is essential that the solution of all these problems is found in a unified way, namely by reducing them to a problem of linear relationship between the boundary values of analytic functions (N. I. Muskhelishvili's method). An account of the available results in this direction with detailed references may be found in the cited monograph of N. I. Muskhelishvili [93].

In other cases, when the segment of contact is obviously not small or straight, no solution of closed form is to be expected. This is so, for instance, in problems of contact between two cylindrical bodies inserted into each other when their cross sections are the same or only slightly different. An additional characteristic feature of these problems is that for surfaces with no corners the region of contact is not given beforehand.

For the case of circular cross sections problems of this kind lead to singular integro-differential equations of a special type or to singular integral equations, not characteristic, with an unknown line of integration. It is somewhat different with contact between cylindrical bodies with a given segment of contact. In this case the problem is reduced to a singular integral equation of the first kind (with a given line of integration).

For all functional equations encountered here there are at present methods of solution that are simple, and at the same time very convenient for computation, and which furnish the means of finding the solution of the contact problem to any degree of approximation. This will be shown in the present chapter.*

I. APPROXIMATE SOLUTION OF SOME SINGULAR EQUATIONS

Most problems of the theory of elasticity (particularly problems of the mixed type) lead as a rule to singular integral equations containing integrals which are interpreted as the Cauchy principal value. Singular equations are usually investigated by reducing them to Fredholm-type regular equations equivalent to the former in the sense of solvability in a given class. For practical purposes, however, this reduction can hardly be considered justified because although the kernels are made smooth by regularization, their general structure deteriorates sharply.

In the solution of problems of particular interest for applications, it is therefore advisable to seek direct methods of solution of singular equations which do not require their preliminary reduction to Fredholm's.

The following two examples will serve, in our opinion, as the best illustration of the above.**

11. Solution of a Singular Integro-differential Equation (Multhopp's Method)

Below we shall deal with an equation of the form

$$\frac{\Gamma(x)}{B(x)}-\frac{1}{2\pi}\int_{-1}^{1}\frac{\Gamma'(t)\,dt}{t-x}=f(x), \qquad \Gamma'(t)=\frac{d\Gamma(t)}{dt}, \tag{11.1}$$

where $B(x)$, $f(x)$ are given functions on the interval $[-1, 1]$ and $\Gamma(x)$ is the unknown function.

In the following it will be assumed that $B(x)$ and $f(x)$ satisfy the Hölder condition on $[-1, 1]$ and that $B(x)$ nowhere vanishes, with the possible exception of the ends; in the case when $B(x)$ is a negative function it is supposed that $B(x) \neq 0$ everywhere on $[-1, 1]$. We require of the desired solution that the derivative $\Gamma'(x)$ should belong to the class H^*, or more exactly that it should satisfy the Hölder condition on every closed part of the interval $[-1, 1]$, not containing the ends, and that near the ends it should admit the representation

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* The method presented here for the solution of contact problems may be generalized to the case of non-circular sections of bodies in contact, a consideration which will be developed later.

** The present section reproduces with slight modifications the author's papers [40, 43].

$$\Gamma'(x)=\frac{\Gamma_0(x)}{(1-x^2)^\alpha}, \qquad 0<\alpha<1, \tag{11.2}$$

where $\Gamma_0(x)$ satisfies the Hölder condition. In addition, it is required that

$$\Gamma(1)=\Gamma(-1)=0. \tag{11.3}$$

The well-known equation of finite wing theory represents an equation of the form (11.1), the function $B(x)$ being non-negative in this case according to its mechanical meaning; $B(x)>0$ on $(-1, 1)$.

The number of works devoted to the equation of wing theory is enormous.* Below we shall be concerned with an approximate method of its solution due to Multhopp [87]. A brief account of the method is given in what follows.

Making the substitution** $x=\cos\vartheta$ $(t=\cos\tau)$ in Eq. (11.1), we write this equation in the form (we again set $\Gamma(\cos\vartheta)=\Gamma(\vartheta), \ldots$)

$$\frac{\Gamma(\vartheta)}{B(\vartheta)}+\frac{1}{2\pi}\int_0^\pi \frac{\Gamma'(\tau)\,d\tau}{\cos\tau-\cos\vartheta}=f(\vartheta) \qquad \left(\Gamma'(\tau)=\frac{d\Gamma(\tau)}{d\tau}\right). \tag{11.4}$$

Let us construct a Lagrange interpolation (trigonometric) polynomial for the unknown function $\Gamma(\vartheta)$. For this purpose, we assign a natural number n and, according to representation (11.2), take as interpolation nodes on $(-1, 1)$ the roots of the Chebyshev polynomial of the second kind of degree n, i.e., the points

$$x_k=\cos\vartheta_k, \quad \vartheta_k=\frac{k\pi}{n+1} \qquad (k=1, 2, \ldots, n). \tag{11.5}$$

The (odd) trigonometric polynomial of degree not greater than n, coincident with $\Gamma(\vartheta)$ at the points ϑ_k, is then written as***

$$L[\Gamma;\, x]=\frac{1}{n+1}\sum_{k=1}^{n}\Gamma(x_k)\frac{(-1)^{k+1}\sin\vartheta_k\sin(n+1)\vartheta}{\cos\vartheta-\cos\vartheta_k} \tag{11.6}$$

or, which is the same thing,

$$L[\Gamma;\, x]=\frac{2}{n+1}\sum_{k=1}^{n}\Gamma(x_k)\sum_{m=1}^{n}\sin m\vartheta_k\sin m\vartheta \qquad (x=\cos\vartheta). \tag{11.7}$$

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* Some references may be found, for example, in the book by V. V. Golubev [29]; see also N. I. Muskhelishvili [92].

** The argument ϑ varies in the range $[0, \pi]$. The same letter ϑ is used throughout to denote the polar co-ordinate of a point ζ in the plane, $\zeta=\rho e^{i\vartheta}$ But this should not cause any confusion.

*** Note that $L[\Gamma;\, x]$ as a function of x, has the form $\sqrt{1-x^2}\,P_{n-1}(x)$, where $P_{n-1}(x)$ is a polynomial in x of degree not greater than $n-1$. Obviously,

$$\Gamma(x_k)=\sqrt{1-x_k^2}\,P_{n-1}(x_k) \quad (k=1, 2, \ldots, n).$$

By differentiating expression (11.7) with respect to ϑ and using the familiar equalities

$$\frac{1}{\pi}\int_0^\pi \frac{\cos m\tau\, d\tau}{\cos\tau-\cos\vartheta}=\frac{\sin m\vartheta}{\sin\vartheta} \qquad (m=0,\ 1,\ \ldots;\ 0\leqslant\vartheta\leqslant\pi),$$

we obtain the formula

$$\frac{1}{2\pi}\int_0^\pi \Gamma'(\tau)\frac{d\tau}{\cos\tau-\cos\vartheta}\cong\frac{1}{2\pi}\int_0^\pi \frac{dL[\Gamma;\ t]}{d\tau}\,\frac{d\tau}{\cos\tau-\cos\vartheta}=$$

$$=\frac{1}{n+1}\sum_{k=1}^{n}\Gamma(x_k)\sum_{m=1}^{n}\frac{m\sin m\vartheta_k\sin m\vartheta}{\sin\vartheta} \qquad (11.8)$$

which will be taken as the mechanical quadrature formula for our improper integral. It is evident that this formula is exact whenever $\Gamma(\vartheta)$ is an odd trigonometric polynomial of degree not higher than n, i.e., when $\Gamma(\vartheta)=\sum\limits_{k=1}^{n} c_k\sin k\vartheta$.

By replacing the integral term in Eq. (11.4) by expression (11.8) and assigning to the argument ϑ successively the values ϑ_1, ϑ_2, ..., ϑ_n, we obtain, for determining approximate values of the unknown function at the points ϑ_k (denote them by $\overline{\Gamma}_k$), the system of linear equations

$$b_\nu\overline{\Gamma}_\nu=f_\nu+\sum_{k=1}^{n}{}' b_{\nu k}\overline{\Gamma}_k \qquad (\nu=1,\ 2,\ \ldots,\ n), \qquad (11.9)$$

where

$$\left.\begin{aligned}
&b_\nu=\frac{1}{B_\nu}+b_{\nu\nu},\qquad b_{\nu\nu}=\frac{n+1}{4\sin\vartheta_\nu},\qquad b_{\nu k}=0\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{when } |\nu-k|=2,\ 4,\ \ldots,\\
&b_{\nu k}=\frac{1}{4(n+1)\sin\vartheta_\nu}\left[\frac{1}{\tan^2\frac{\vartheta_k-\vartheta_\nu}{2}}-\frac{1}{\tan^2\frac{\vartheta_k+\vartheta_\nu}{2}}\right]\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{when } |\nu-k|=1,\ 3,\ \ldots,\\
&B_\nu=B(\vartheta_\nu),\qquad f_\nu=f(\vartheta_\nu)\qquad (\nu=1,\ 2,\ \ldots,\ n).
\end{aligned}\right\} \qquad (11.10)$$

If the solution $\overline{\Gamma}_k$ of system (11.9) is substituted for Γ_k in expression (11.6), we obtain the function

$$\overline{\Gamma}_n(x)=\frac{1}{n+1}\sum_{k=1}^{n}(-1)^{k+1}\overline{\Gamma}_k\frac{\sin(n+1)\vartheta\sin\vartheta_k}{\cos\vartheta-\cos\vartheta_k} \qquad (11.11)$$

which must serve as an approximate solution of Eq. (11.1).

In view of the fact that $b_{\nu k} = 0$ when $|\nu - k| = 2, 4, \ldots$ $(k \leqslant n;\ \nu \leqslant n)$, system (11.9) separates; more precisely, the unknowns with odd subscripts in this system are expressed solely in terms of the unknowns with even subscripts and vice versa. Owing to this and some other properties of the coefficients $b_{\nu k}$, the method of successive approximations appears to be best suited here.* System (11.9) is solved in practice by the above-mentioned method. For the purpose of a suitable selection of the initial approximation, the only values considered are $n = 2^m - 1$ $(m = 2, 3, \ldots)$. In the symmetrical case when $B(x) = B(-x)$, $f(x) = f(-x)$, and consequently $\Gamma(x) = \Gamma(-x)$, the foregoing scheme is considerably simplified; in this case $\overline{\Gamma}_k = \overline{\Gamma}_{n+1-k}$ $\left(k = 1, 2, \ldots, \left[\frac{n+1}{2}\right]\right)$, and hence system (11.9) represents a system of only $\left[\frac{n+1}{2}\right]$ equations.**

It appears from the above that Multhopp's method represents a scheme for numerical solution of Eq. (11.1) that is at once very simple and convenient for practical applications. Besides, as a closer check shows, the desired accuracy can be achieved at the very first steps.***

In the next section we shall take up a justification of this method. Under certain conditions which will be stated below we shall prove that the iterative process as applied to system (11.9) converges, and the sequence $\overline{\Gamma}_n(x)$ also converges uniformly to the solution of Eq. (11.1).

12. Investigation of the Method

1°. We introduce into consideration a linear space $\overline{X}$ of n-dimensional vectors $\xi = (\xi_1, \ldots, \xi_n)$, where the norm is defined as follows:

$$\|\xi\| = \max_k |\xi_k|.$$

System (11.9) is written in the form of a functional equation

$$\overline{K}_0\overline{\Gamma} \equiv \overline{\Gamma} - \overline{H}_0\overline{\Gamma} = \overline{f}_0, \tag{12.1}$$

where

$$\overline{\Gamma} = (\overline{\Gamma}_1, \ldots, \overline{\Gamma}_n), \qquad \overline{f}_0 = \left(\frac{f_1}{b_1}, \ldots, \frac{f_n}{b_n}\right)$$

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* In the case of the equation of wing theory it leads to the purpose much more quickly.

** Here, as before, the square brackets denote the integral part of the number inside.

*** In practical applications of the method, the three commonly taken values for the index n are 7, 15 and 31. The approximate solutions $\overline{\Gamma}_n(x)$ corresponding to these values are customarily called the first, second and third approximations. Provided the given functions (the chord of an airfoil and the geometrical angle of attack) are smooth, even the first approximation is found to be quite sufficient in practice (see the cited work of Multhopp, p. 157).

and $\bar{H}_0$ is a linear operation transforming X into $\bar{X}$ and given by the matrix $\left(\frac{b_{\nu k}}{b_\nu}\right)$.

As is known, the question of interest to us leads to an estimate of the norm (assuming that $b_\nu > 0$)

$$\|\bar{H}_0\| = \max_\nu \sum_{k=1}^{n} \frac{b_{\nu k}}{b_\nu},$$

which, on the basis of (11.10), is represented as*

$$\|\bar{H}_0\| = \max_\nu \frac{B_\nu}{(n+1)\,[(n+1)\,B_\nu + 4\sin\vartheta_\nu]} \times$$

$$\times \sum_{k=1}^{n} \left[\frac{1}{\tan^2 \frac{\vartheta_k - \vartheta_\nu}{2}} - \frac{1}{\tan^2 \frac{\vartheta_k + \vartheta_\nu}{2}}\right]. \tag{12.2}$$

As can easily be observed

$$\frac{B_\nu}{(n+1)\,[(n+1)\,B_\nu + 4\sin\vartheta_\nu]} \leqslant \frac{1}{1 - \frac{4\lambda}{n+1}} \frac{1}{(n+1)^2},$$

where

$$\lambda = \begin{cases} \max\limits_{x \in [-1,\ 1]} \left|\frac{1}{B(x)}\right| & \text{when } B(x) < 0, \\ 0 & \text{when } B(x) > 0, \end{cases} \tag{12.3}$$

and the equality may be attained [in the case of negative $B(x)$] when $\vartheta_\nu = \pi/2$. By virtue of the above estimate, from formula (12.2) we have

$$\|\bar{H}_0\| \leqslant \frac{1}{1 - \frac{4\lambda}{n+1}} \frac{1}{(n+1)^2} \max_\nu \times$$

$$\times \sum_{k=1}^{n} \left[\frac{1}{\tan^2 \frac{\vartheta_k - \vartheta_\nu}{2}} - \frac{1}{\tan^2 \frac{\vartheta_k + \vartheta_\nu}{2}}\right]. \tag{12.4}$$

We set

$$\left.\begin{aligned} \Omega_1(\nu) &= \frac{1}{(n+1)^2} \sum_{k=1}^{n} \frac{1}{\tan^2 \frac{\vartheta_k - \vartheta_\nu}{2}}, \\ \Omega_2(\nu) &= \frac{1}{(n+1)^2} \sum_{k=1}^{n} \frac{1}{\tan^2 \frac{\vartheta_k + \vartheta_\nu}{2}}. \end{aligned}\right\} \tag{12.5}$$

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* In both this and the following sums involving two indices, k and ν, the summation index runs from 1 to n so that the number $|k - \nu|$ remains odd.

Taking into account formula (11.5), we represent Ω_1 in the form

$$\Omega_1(\nu)=\frac{4}{\pi^2}\sum_{k=1}^{n}\frac{1}{(k-\nu)^2}-\frac{1}{(n+1)^2}\sum_{k=1}^{n}\left[\frac{1}{\left(\frac{\vartheta_k-\vartheta_\nu}{2}\right)^2}-\frac{1}{\tan^2\frac{\vartheta_k-\vartheta_\nu}{2}}\right].$$

The last equality is rewritten as

$$\Omega_1(\nu)=\frac{4}{\pi^2}\sum_{k=1}^{n}\frac{1}{(k-\nu)^2}-\frac{1}{2\pi(n+1)}\times$$

$$\times\int_0^{\pi}\left[\frac{1}{\left(\frac{\vartheta-\vartheta_\nu}{2}\right)^2}-\frac{1}{\tan^2\frac{\vartheta-\vartheta_\nu}{2}}\right]d\vartheta+\frac{J(\nu)}{n+1}, \qquad (12.6)$$

$$J(\nu)=\frac{1}{2\pi}\left\{\int_0^{\pi}\left[\frac{1}{\left(\frac{\vartheta-\vartheta_\nu}{2}\right)^2}-\frac{1}{\tan^2\frac{\vartheta-\vartheta_\nu}{2}}\right]d\vartheta-\right.$$

$$\left.-\frac{2\pi}{n+1}\sum_{k=1}^{n}\left[\frac{1}{\left(\frac{\vartheta_k-\vartheta_\nu}{2}\right)^2}-\frac{1}{\tan^2\frac{\vartheta_k-\vartheta_\nu}{2}}\right]\right\}. \qquad (12.7)$$

It is easy to obtain the estimate

$$\sum_{k=1}^{n}\frac{1}{(k-\nu)^2}\leqslant 2\sum_{k=1,\,3,\,\dots}^{\left[\frac{n+1}{2}\right]}\frac{1}{k^2};$$

the equality sign holds when $n=4m-1$, $\nu=2m$ $\left(\vartheta_\nu=\frac{\pi}{2}\right)$ or when $n=4m$, $\nu=2m$, $2m+1$. But

$$\sum_{k=1,\,3,\,\dots}^{\left[\frac{n+1}{2}\right]}\frac{1}{k^2}=\sum_{k=1,\,3,\,\dots}^{\infty}\frac{1}{k^2}-\sum_{2m+1=k\geqslant\left[\frac{n+1}{2}\right]+1}^{\infty}\frac{1}{k^2}=$$

$$=\frac{\pi^2}{8}-\sum_{2m+1=k\geqslant\left[\frac{n+1}{2}\right]+1}^{\infty}\frac{1}{k^2}.$$

On the other hand, by virtue of the inequality

$$\frac{1}{2n-1}<2\sum_{k=n}^{\infty}\frac{1}{(2k-1)^2}<\frac{1}{2n-3}$$

we have

$$\sum_{2m+1=k\geqslant\left[\frac{n+1}{2}\right]+1}^{\infty}\frac{1}{k^2}>\frac{1}{2}\,\frac{1}{\left[\frac{n+1}{2}\right]+2}\geqslant\frac{1}{n+5}.$$

Consequently, for the first term on the right-hand side of equality (12.6) we have, finally,

$$\frac{4}{\pi^2}\sum_{k=1}^{n}\frac{1}{(k-\nu)^2}<1-\frac{8}{\pi^2(n+5)}. \qquad (12.8)$$

We now turn to the second term in equality (12.6). Taking the integral, we find

$$\frac{1}{2\pi}\int_0^{\pi}\left[\frac{1}{\left(\frac{\vartheta-\vartheta_\nu}{2}\right)^2}-\frac{1}{\tan^2\frac{\vartheta-\vartheta_\nu}{2}}\right]d\vartheta=$$
$$=\frac{2}{\pi}\left[\frac{1}{\sin\vartheta_\nu}-\frac{1}{\pi-\vartheta_\nu}-\frac{1}{\vartheta_\nu}+\frac{\pi}{4}\right].$$

The function

$$F(\vartheta)=\frac{1}{\sin\vartheta}-\frac{1}{\pi-\vartheta}-\frac{1}{\vartheta}+\frac{\pi}{4}$$

increases on $\left(0,\frac{\pi}{2}\right)$, attains extreme values at the points $\vartheta=0,\frac{\pi}{2}$, and its fluctuation is insignificant. We have

$$\max F(\vartheta)=F\left(\frac{\pi}{2}\right)=1+\frac{\pi}{4}-\frac{4}{\pi},$$
$$\min F(\vartheta)=F(0)=\frac{\pi}{4}-\frac{1}{\pi},$$
$$F(\vartheta)=F(\pi-\vartheta)\qquad\left(0\leqslant\vartheta\leqslant\frac{\pi}{2}\right).$$

Consequently,

$$\frac{1}{2}-\frac{2}{\pi^2}\leqslant\frac{1}{2\pi}\int_0^{\pi}\left[\frac{1}{\left(\frac{\vartheta-\vartheta_\nu}{2}\right)^2}-\frac{1}{\tan^2\frac{\vartheta-\vartheta_\nu}{2}}\right]d\vartheta\leqslant\frac{2}{\pi}+\frac{1}{2}-\frac{8}{n^2}. \qquad (12.9)$$

The equality on the right-hand side is attained for $\vartheta_\nu=\frac{\pi}{2}$, which is possible for odd n.

Furthermore, it can readily be verified that for the remaining term (12.7), which represents (possibly, apart from a term of order $1/n$) the difference between the integral of a continuous function and the corresponding Riemann sum, the following estimate is valid:

$$J(\nu)<\frac{1}{n+1}. \qquad (12.10)$$

By virtue of the foregoing estimates we have

$$\Omega_1(\nu) < 1 - \left(\frac{2}{\pi} + \frac{1}{2}\right)\frac{1}{n+1} + \frac{32}{\pi^2}\,\frac{1}{(n+1)(n+5)} + \frac{1}{(n+1)^2}. \qquad (12.11)$$

We proceed to the estimation of $\Omega_2(\nu)$. Consider the functions

$$p_k(\vartheta) = \cot^2\frac{\vartheta_k+\vartheta}{2} + \cot^2\frac{\pi-\vartheta_k+\vartheta}{2},$$

$$q_k(\vartheta) = \cot^2\frac{\vartheta_k+\vartheta}{2} + \cot^2\frac{\pi-\vartheta_k+\dfrac{(-1)^k\pi}{n+1}+\vartheta}{2} \qquad (k=1, 2, \ldots, n).$$

By direct checking it may be verified that

$$\min_{\vartheta\in[0,\pi]} p_k(\vartheta) = p_k\left(\frac{\pi}{2}\right), \qquad \min_{\vartheta\in[0,\pi]} q_k(\vartheta) = q_k\left(\frac{\pi}{2} + \frac{(-1)^{k-1}\pi}{2(n+1)}\right).$$

The above equalities enable us to conclude that

$$\sum_{k=1}^{n} \cot^2\frac{\vartheta_k+\vartheta_\nu}{2} \geqslant \sum_{k=1}^{n} \cot^2\frac{\vartheta_k+\vartheta^*}{2}, \qquad (12.12)$$

where

$$\vartheta^* = \begin{cases} \dfrac{\pi}{2} & \text{if } n \text{ is odd,} \\ \dfrac{\pi}{2} + \dfrac{(-1)^\nu\pi}{2(n+1)} & \text{if } n \text{ is even.} \end{cases} \qquad (12.13)$$

In carrying out the summation on the right-hand side of inequality (12.12), the index k takes even or odd values from 1 to n according as ν is odd or even. The equality in (12.12) is always realizable except in the case of $n = 4m$. The right-hand side of inequality (12.12) is represented in the form

$$\frac{2\pi}{n+1}\sum_{k=1}^{n} \cot^2\frac{\vartheta_k+\vartheta^*}{2} = \int_0^\pi \cot^2\frac{\vartheta+\vartheta^*}{2}\,d\vartheta - J^*(\vartheta^*), \qquad (12.14)$$

where

$$J^*(\vartheta^*) = \int_0^\pi \cot^2\frac{\vartheta+\vartheta^*}{2}\,d\vartheta - \frac{2\pi}{n+1}\sum_{k=1}^{n} \cot^2\frac{\vartheta_k+\vartheta^*}{2}.$$

$J^*(\vartheta^*)$ may be estimated as

$$J^*(\vartheta^*) \leqslant \int_0^{\vartheta_2} \cot^2\frac{\vartheta+\vartheta^*}{2}\,d\vartheta + \int_{\vartheta_{n-1}}^{\pi} \cot^2\frac{\vartheta+\vartheta^*}{2}\,d\vartheta \leqslant$$

$$\leqslant \vartheta_2\cot^2\frac{\vartheta^*}{2} + (\pi-\vartheta_{n-1})\tan^2\frac{\vartheta^*}{2}.$$

By using here the inequalities $x < \tan x < \frac{2}{\pi - 2x}$ $\left(0 < x < \frac{\pi}{2}\right)$ and taking into account formulas (11.5) and (12.13), we obtain

$$J^*(\vartheta^*) < \frac{64}{\pi}\frac{n+1}{n^2}. \qquad (12.15)$$

Noting, finally, that

$$\int_0^\pi \cot^2 \frac{\vartheta + \vartheta^*}{2} = \frac{4}{\sin \vartheta^*} - \pi \geqslant 4 - \pi,$$

by formulas (12.12), (12.14), (12.15) and (12.5) we have for $\Omega_2(\nu)$ the estimate

$$\Omega_2(\nu) > \left(\frac{2}{\pi} - \frac{1}{2}\right)\frac{1}{n+1} - \frac{32}{\pi^2}\frac{1}{n^2}. \qquad (12.16)$$

Inequalities (12.11) and (12.16) now give, on the basis of inequality (12.4), the following estimate for $\| \overline{H}_0 \|$ $(n > 1)$:

$$\| \overline{H}_0 \| < \frac{1}{1 - \frac{4\lambda}{n+1}}\left[1 - \frac{4}{\pi}\frac{1}{n+1} + \left(\frac{64}{\pi^2} + 1\right)\frac{1}{n(n+1)}\right]. \qquad (12.17)$$

From this it follows that if

$$\lambda = \max_{x \in [-1,\,1]} \left| \frac{1}{B(x)} \right| < \frac{1}{\pi}, \qquad (12.18)$$

then, beginning with some n, $\| \overline{H}_0 \| < 1$. Hence, by the well-known theorem of functional analysis (see, for example, L. V. Kantorovich [56], p. 102), the operator $\overline{K}_0$ has the inverse operator $\overline{K}_0^{-1}$, with

$$\| \overline{K}_0^{-1} \| < \frac{1}{1-q}, \qquad (12.19)$$

where q stands for the right-hand side of inequality (12.17), and the process of successive approximations for Eq. (12.1) converges to the solution of this equation, whatever the initial values from which we proceed. In other words, if condition (12.18) is fulfilled, system (11.9) has a (single) solution which can be found by the method of successive approximations.

Note 1. The result stated above admits no improvement.

Indeed, when $n = 2m + 1$, $\nu = m + 1$ we have the following asymptotic representation:

$$\Omega_1(\nu) - \Omega_2(\nu) = 1 - \frac{4}{\pi}\frac{1}{n} + O\left(\frac{1}{n^2}\right).$$

On the other hand, if the function $B(x)$ is negative and the modulus of the ratio $\left| \frac{1}{B(x)} \right|$ attains its maximum value at the point $x = 0$, then under the same conditions and for sufficiently large n the following equality is valid:

$$\sum_{k=1}^{n} \frac{b_{\nu k}}{b_\nu} = \frac{1}{1-\frac{4\lambda}{n}}\left[1-\frac{4}{\pi}\frac{1}{n}+O\left(\frac{1}{n^2}\right)\right].$$

In order that this quantity be less than unity (beginning with some n), it is just necessary that condition (12.18) should be fulfilled.

Note 2. If we restrict our attention to the case of positive $B(x)$, it is not difficult to prove the convergence of the above process. Indeed, in this case $\lambda = 0$, and hence, rejecting further the term $\Omega_2(\nu)$, we have, from inequality (12.4),

$$\|\overline{H}_0\| < \frac{1}{n+1} \max_\nu \sum_{k=1}^{n} \frac{1}{\tan^2 \frac{\vartheta_k - \vartheta_\nu}{2}}.$$

By using here the inequality $\tan x > x \left(0 < x \leqslant \frac{\pi}{2}\right)$ and then substituting $\vartheta_k = \frac{k\pi}{n+1}$, we obtain

$$\|\overline{H}_0\| < \frac{1}{(n+1)^2} \max_\nu \sum_{k=1}^{n} \frac{4}{(\vartheta_k - \vartheta_\nu)^2} =$$

$$= \frac{4}{\pi^2} \max_\nu \sum_{k=1}^{n} \frac{1}{(k-\nu)^2} < \frac{8}{\pi^2} \sum_{k=1,\,3,\,\ldots}^{\infty} \frac{1}{k^2} = 1.$$

This is done in the work of H. Multhopp [87].

As seen from the above, on the basis of such an estimate of the right-hand side of inequality (12.4) the convergence of the process in the case of negative $B(x)$ may be stated only if $\lambda < \frac{2}{\pi^2}$.

System (11.9) may also be written as

$$\overline{K}\overline{\Gamma} \equiv \overline{A}\,\overline{\Gamma} - \overline{H}\overline{\Gamma} = \overline{f}, \tag{12.20}$$

where $\overline{f} = (f_1, \ldots, f_n)$; $\overline{H}$ and $\overline{A}$ designate operations specified, respectively, by the matrices $(b_{\nu k})$ and $(\alpha_{\nu k})$, with $\alpha_{\nu\nu} = b_\nu$, $\alpha_{\nu k} = 0$ for $\nu \neq k$. But, as proved above, there exists an inverse operation $\overline{K}^{-1}$ and, obviously, $\overline{K}^{-1}\overline{f} = \overline{K}_0^{-1}\overline{f}_0$. On the basis of this equality and relation (11.10) we have

$$\|\overline{K}^{-1}\overline{f}\| < \frac{\|\overline{K}_0^{-1}\overline{f}\|}{\min\limits_\nu b_\nu} \leqslant \frac{4\|\overline{K}_0^{-1}\overline{f}\|}{n+1-4\lambda}. \tag{12.21}$$

On the other hand, according to estimate (12.19) we also have:* $\| \overline{K}_0^{-1} \| \leqslant A_0 n$, A_0 = constant. From formula (12.21) we obtain, therefore,

$$\| \overline{K}^{-1} \overline{f} \| \leqslant \frac{4n}{n+1-4\lambda} A_0 \| \overline{f} \|.$$

From this it follows that

$$\| \overline{K}^{-1} \| \leqslant A^*, \qquad A^* = \text{constant}. \tag{12.22}$$

Consequently, the norms $\| \overline{K}^{-1} \|$ are bounded.

2°. In the following discussion we shall need some propositions from the theory of uniform approximation of functions assigned on an interval. We formulate them as lemmas.

Let the class of all functions $f(x)$ assigned on $[-1, 1]$ and satisfying a Hölder condition with a given exponent α $(0 < \alpha \leqslant 1)$ be denoted by Lip α. The least Hölder's constant for $f(x)$ with a given α is denoted by $M[f; \alpha]$, i.e., by definition,

$$M[f; \alpha] = \sup_{x_1, x_2 \in [-1, 1]} \frac{|f(x_2) - f(x_1)|}{|x_2 - x_1|^{\alpha}}. \tag{12.23}$$

Lemma 1. *Let n be a natural number and $\vartheta_k = \frac{k\pi}{n+1}$. Assume* $(x = \cos\vartheta)$

$$\lambda_n(x) = \frac{1}{n+1} \sum_{k=1}^{n} \left| \frac{\sin(n+1)\vartheta}{\cos\vartheta - \cos\vartheta_k} \right| \sin\vartheta_k.$$

Then uniformly on the interval $[-1, 1]$

$$\lambda_n(x) \leqslant 8 + \frac{4}{\pi} \ln(n+1). \tag{12.24}$$

This lemma is proved in exactly the same way as the well-known theorem of S. N. Bernshtein, giving the estimate**

$$\frac{1}{n} \sum_{k=1}^{n} \left| \frac{\cos n\vartheta}{\cos\vartheta - \cos\vartheta_k'} \right| \sin\vartheta_k' \leqslant 8 + \frac{4}{\pi} \ln n, \quad \vartheta_k' = \frac{(2k-1)\pi}{2n}.$$

Lemma 2. *Let a function $f(x)$ of the class* Lip α *be given. Let for every natural n there be an algebraic polynomial $p_n(x)$ for which*

$$|f(x) - p_n(x)| < \frac{A_1}{n^{\alpha}} \qquad (A_1 = \text{constant}). \tag{12.25}$$

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* If condition (12.18) is fulfilled, the right-hand side of inequality (12.17) is of the form

$$q = 1 - \frac{c}{n} + O\left(\frac{1}{n^2}\right), \quad c > 0.$$

** For a proof of the theorem of S. N. Bernshtein see, for example, the book by I. P. Natanson [96], pp. 539-542.

Then the following estimate is valid:

$$M[f_n;\beta] \leqslant \frac{A_2}{n^{\alpha-2\beta}}, \tag{12.26}$$

where $f_n(x) = f(x) - p_n(x)$, β *is any positive number such that* $2\beta < \alpha$, A_2 *is a constant depending on* α *and* β.

Proof. Let us fix a natural number n and, on the basis of inequality (12.25), set (S. N. Bernshtein [6]; see also S. B. Stechkin [130])

$$f_n(x) = f(x) - p_n(x) = \sum_{k=1}^{\infty} U_k(x), \tag{12.27}$$

where

$$U_k(x) = p_{2^k n}(x) - p_{2^{k-1}n}(x).$$

Let us estimate the polynomial $U_k(x)$. By inequality (12.25) we have

$$|U_k(x)| \leqslant |p_{2^k n}(x) - f(x)| + |p_{2^{k-1}n}(x) - f(x)| \leqslant$$

$$\leqslant \frac{A_1}{(2^k n)^\alpha} + \frac{A_1}{(2^{k-1}n)^\alpha} = \frac{A_k^*}{(2^{k-1}n)^{2\beta}}, \tag{12.28}$$

where

$$A_k^* = \frac{(1+2^{-\alpha})A_1}{(2^{k-1}n)^{\alpha-2\beta}}. \tag{12.29}$$

Assign an arbitrary positive number h, $0 < h \leqslant 2$. To begin with, we assume that $4n^2h \leqslant 1$ and select a natural number ν from the condition

$$2^{-2(\nu+1)}n^{-2} < h \leqslant 2^{-2\nu}n^{-2}. \tag{12.30}$$

Accordingly we represent formula (12.27) as

$$f_n(x) = F_1(x) + F_2(x), \tag{12.31}$$

where

$$F_1(x) = \sum_{k=1}^{\nu} U_k(x), \qquad F_2(x) = \sum_{k=\nu+1}^{\infty} U_k(x).$$

It is evident that

$$|\Delta_h F_1(x)| \leqslant \sum_{k=1}^{\nu} |\Delta_h U_k(x)|,$$

where use has been made of the commonly employed notation

$$\Delta_h f(x) = f(x+h) - f(x).$$

By applying first to each term on the right-hand side Lagrange's mean value theorem and then the inequality of A. A. Markov,*

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* The well-known inequality of A. A. Markov (see, for example, I. P. Natanson [96], p. 174) states the following: if $P(x)$ is a polynomial of degree n in x and M is the maximum of its modulus on $[-1, 1]$, then uniformly on this interval

$$|P'(x)| \leqslant Mn^2.$$

and using inequality (12.28), we have

$$|\Delta_h F_1(x)| \leqslant \sum_{k=1}^{\nu} h\,|U_k'(x+\theta_k h)| \leqslant h \sum_{k=1}^{\nu} (2^k n)^2 \max_x |U_k(x)| \leqslant$$

$$\leqslant 2^{2\beta} h \sum_{k=1}^{\nu} (2^k n)^{2(1-\beta)} A_k^* \leqslant 2^{2\beta} A_1^* h\,(2^\nu n)^{2(1-\beta)} \sum_{k=0}^{\nu-1} 2^{-2(1-\beta)k}. \qquad (12.32)$$

But by virtue of the second part of inequality (12.30)

$$(2^\nu n)^{2(1-\beta)} \leqslant h^{\beta-1}.$$

Moreover,

$$\sum_{k=0}^{\nu-1} 2^{-2(1-\beta)k} \leqslant \frac{2^{2(1-\beta)}}{2^{2(1-\beta)}-1}.$$

On the basis of the last relations and formula (12.29) (with $k = 1$) inequality (12.32) takes the form

$$|\Delta_h F_1(x)| < \frac{4\,(1+2^{-\alpha})\,A_1 h^\beta}{(2^{2(1-\beta)}-1)\,n^{\alpha-2\beta}}. \qquad (12.33)$$

Further we have, by formula (12.28),

$$|\Delta_h F_2(x)| \leqslant 2 \sum_{k=\nu+1}^{\infty} \max_x |U_k(x)| \leqslant 2A_{\nu+1}^* \sum_{k=\nu+1}^{\infty} (2^{k-1} n)^{-2\beta} =$$

$$= 2^{2\beta+1} A_{\nu+1}^* (2^{\nu+1} n)^{-2\beta} \sum_{k=0}^{\infty} 2^{-2\beta k}. \qquad (12.34)$$

But according to the left inequality of (12.30) $(2^{\nu+1} n)^{-2\beta} < h^\beta$. From formula (12.34), with (12.29), we obtain, therefore,

$$|\Delta_h F_2(x)| < \frac{2^{4\beta+1}\,(1+2^{-\alpha})}{2^{2\beta}-1}\,\frac{A_1 h^\beta}{n^{\alpha-2\beta}}. \qquad (12.35)$$

Let now $4n^2 h > 1$. By virtue of this inequality and estimate (12.25) we then have

$$\frac{|\Delta_h f_n(x)|}{h^\beta} \leqslant 2\,(4n^2)^\beta \max_x |f_n(x)| \leqslant \frac{2^{2\beta+1} A_1}{n^{\alpha-2\beta}}$$

and hence

$$|\Delta_h f_n(x)| \leqslant \frac{2^{2\beta+1} A_1}{n^{\alpha-2\beta}}\, h^\beta. \qquad (12.36)$$

From relations (12.33), (12.35) and (12.36) the estimate (12.26) is obtained, and for A_2 we find: $A_2 = A_1 A_3$, where

$$A_3 = \max\left\{ 2^{2\beta+1},\ \frac{4\,(1+2^{-\alpha})}{2^{2(1-\beta)}-1},\ \frac{2^{4\beta+1}\,(1+2^{-\alpha})}{2^{2\beta}-1} \right\}.$$

The lemma is thus proved.*

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* We are grateful to S. M. Nikol'skii whose comments were used in the proof of the present lemma.

Lemma 3 (A. O. Gel'fond [26], p. 50). *Let a function $f(x)$ assigned on $[-1, 1]$ have a derivative of order m, with $f^m(x) \in \operatorname{Lip}\alpha$ $(0 < \alpha \leqslant 1)$. Then for every natural n there exists a polynomial $p_n(x)$ of degree not higher than n for which*

$$|f^{(\nu)}(x) - p_n^{(\nu)}(x)| \leqslant \frac{A_4}{n^{m-\nu+\alpha}} \qquad (\nu = 0, 1, \ldots, m; \ -1 \leqslant x \leqslant 1), \tag{12.37}$$

where A_4 is a constant (depending on f and m).

3°. We now turn to the question of convergence of sequence (11.11). Returning to Eq. (11.1), we further assume that the ratio

$$\frac{\sqrt{1-x^2}}{B(x)} \tag{12.38}$$

satisfies the Hölder condition on the interval $[-1, 1]$. It will be understood that Eq. (11.1) has, with given B and f, a solution which satisfies all the conditions stated at the beginning of the preceding section. It may be proved that this solution, $\Gamma(x)$, is representable in the form

$$\Gamma(x) = \sqrt{1-x^2}\,\gamma(x), \tag{12.39}$$

where $\gamma(x)$ satisfies the Hölder condition on $[-1, 1]$. Condition (11.2) imposed above on the solution is now replaced by a stronger one, namely that the derivative of γ in representation (12.39) should belong to the class $\operatorname{Lip}\alpha$ $(0 < \alpha \leqslant 1)$.

Introduce into consideration a linear space X_β of given functions $\Gamma(x)$ on $[-1, 1]$ of the form $\sqrt{1-x^2}\,\gamma(x)$, where $\gamma'(x) \in \operatorname{Lip}\beta$, $0 < \beta \leqslant 1$. The norm of element $\Gamma(x)$ is defined as follows:

$$\|\Gamma\|_\beta = \max_{x \in [-1, 1]} |\gamma(x)| + \max_{x \in [-1, 1]} |\gamma'(x)| + M[\gamma'; \beta], \tag{12.40}$$

where $M[\gamma'; \beta]$ is calculated by formula (12.23).

Equation (11.1) is written as

$$K\Gamma = f. \tag{12.41}$$

It is easily verified that K is a linear operation acting in the space X_β and transforming it into a space of all continuous functions defined on $[-1, 1]$.

Indeed, setting $\Gamma(x) = \sqrt{1-x^2}\gamma(x)$ ($\gamma'(x) \in \operatorname{Lip}\beta$), we represent the left-hand side of Eq. (12.41) in the form*

$$K\Gamma \equiv \frac{\sqrt{1-x^2}}{B(x)}\gamma(x) + \frac{x}{2\pi}\int_{-1}^{1}\frac{\gamma(t)-\gamma(x)}{\sqrt{1-t^2}}\frac{dt}{t-x} + \frac{1}{2\pi}\int_{-1}^{1}\frac{\gamma(t)\,dt}{\sqrt{1-t^2}} - \\ -\frac{\gamma'(x)}{2\pi}\int_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}\,dt - \frac{1}{2\pi}\int_{-1}^{1}\frac{\sqrt{1-t^2}\,[\gamma'(t)-\gamma'(x)]}{t-x}\,dt. \qquad (12.42)$$

The continuity of the above expression follows immediately from the Plemelj-Privalov theorem (see, for example, N. I. Muskhelishvili [92], Sec. 16). Further, taking into account the equalities

$$\frac{1}{\pi}\int_{-1}^{1}\frac{dt}{\sqrt{1-t^2}} = 1, \qquad \frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-t^2}\,dt}{t-x} = -x,$$

from relation (12.42) we obtain**

$$\| K\Gamma \| \leqslant \| K \| \cdot \| \Gamma \|_{\beta} \qquad (12.43)$$

with

$$\| K \| \leqslant \max\left\{\frac{1}{2} + \max_x \frac{\sqrt{1-x^2}}{B(x)},\ 1,\ \max_x \frac{1}{2\pi}\int_{-1}^{1}\frac{\sqrt{1-t^2}\,dt}{|t-x|^{1-\beta}}\right\}. \qquad (12.44)$$

In the following it will be assumed that α and β are fixed positive numbers, with β selected from the condition $2\beta < \alpha$ $(0 < \alpha \leqslant 1)$.

We introduce (for a given n)*** a set Y of all functions of the form $P_n(x) = \sqrt{1-x^2}p_{n-1}(x)$, where $p_{n-1}(x)$ is a polynomial of degree $n-1$ in x, and consider this set as a linear subspace of the space C. Let us define on Y a linear operation φ_0, assuming for $P_n \in Y$

$$\varphi_0 P_n = (P_n(x_1), \ldots, P_n(x_n)), \qquad x_k = \cos\frac{k\pi}{n+1} \qquad (k = 1, 2, \ldots, n).$$

It is apparent that φ_0 effects an isomorphic mapping of Y onto $\overline{X}$, and for the inverse operation we have

$$\varphi_0^{-1}\xi = \frac{1}{n+1}\sum_{k=1}^{n}(-1)^{k+1}\xi_k\left|\frac{\sin(n+1)\vartheta\sin\vartheta_k}{\cos\vartheta - \cos\vartheta_k}\right|,$$

$$\xi = (\xi_1, \ldots, \xi_n) \in \overline{X}.$$

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* Here use has been made of the equality

$$\int_{-1}^{1}\frac{dt}{\sqrt{1-t^2}\,(t-x)} = 0 \qquad (-1 \leqslant x \leqslant 1).$$

** The function $K\Gamma$ is considered to be an element of space C. By $\| K\Gamma \|$ is therefore meant $\max_x | K\Gamma |$ on $[-1, 1]$.

*** Below we use a scheme of L. V. Kantorovich [56] (Chap. II, Sec. 1).

The operation φ_0^{-1} is a linear operation in $\overline{X}$. Indeed,

$$\max_x |\varphi_0^{-1}\xi| \leqslant \frac{1}{n+1} \max_x \sum_{k=1}^{n} \left| \frac{\sin (n+1)\vartheta \sin \vartheta_k}{\cos \vartheta - \cos \vartheta_k} \right| |\xi_k| \leqslant$$

$$\leqslant \max_k |\xi_k| \frac{1}{n+1} \max_x \sum_{k=1}^{n} \left| \frac{\sin (n+1)\vartheta \sin \vartheta}{\cos \vartheta - \cos \vartheta_k} \right|.$$

Hence, by Lemma 1

$$\| \varphi_0^{-1}\xi \| \leqslant \left[8 + \frac{4}{\pi} \ln (n+1) \right] \| \xi \|.$$

Consequently,

$$\| \varphi_0^{-1} \| \leqslant 8 + \frac{4}{\pi} \ln (n+1). \tag{12.45}$$

The operation φ_0 extended to the whole space C is designated by φ. In other words, we define a linear operation φ which maps C onto X and to each continuous function puts into correspondence a set of its values at points x_k; for $y(x) \in C$ we assume

$$\varphi y = (y(x_1), \ldots, y(x_n)).$$

It is obvious that

$$\| \varphi \| = 1. \tag{12.46}$$

Let now $\Gamma(x)$ be a solution of Eq. (12.1), $\Gamma(x) = \sqrt{1-x^2}\gamma(x)$, $\gamma'(x) \in \text{Lip}\,\alpha$. We construct, according to Lemma 3, a sequence of polynomials $\gamma_{n-1}(x)$ for which

$$|\gamma^{(\nu)}(x) - \gamma_{n-1}^{(\nu)}(x)| \leqslant \frac{A_5}{n^{1-\nu+\alpha}} \qquad (\nu = 0, 1). \tag{12.47}$$

Then, on the basis of Lemma 2,

$$M[\psi_n'; \beta] \leqslant \frac{A_6}{n^{\alpha-2\beta}}, \qquad \psi_n(x) = \gamma(x) - \gamma_{n-1}(x).$$

By the definition of the norm (12.40), the above two inequalities give

$$\| \Gamma - \Gamma_n \|_\beta \leqslant \frac{A_7}{n^{\alpha-2\beta}}, \qquad \Gamma_n(x) = \sqrt{1-x^2}\,\gamma_{n-1}(x). \tag{12.48}$$

Let now $\overline{\Gamma}_1$, $\overline{\Gamma}_0$ denote solutions of Eq. (12.20) with the right-hand sides as $\varphi K \Gamma_n$ and $\varphi K \Gamma$, respectively. Hence,

$$\overline{K\Gamma_1} = \varphi K \Gamma_n, \qquad \overline{K\Gamma_0} = \varphi K \Gamma. \tag{12.49}$$

By subtracting these equations member by member, we obtain

$$\overline{K}(\overline{\Gamma}_0 - \overline{\Gamma}_1) = \varphi K (\Gamma - \Gamma_n). \tag{12.50}$$

But, as proved above (see Par. 1°), the operation $\overline{K}^{-1}$ exists. Taking also into account formulas (12.43) and (12.46), from relation (12.50) we obtain, therefore,

$$\|\overline{\Gamma}_0 - \overline{\Gamma}_1\| \leqslant \|\overline{K}^{-1}\|\cdot\|K\|\cdot\|\Gamma - \Gamma_n\|_\beta.$$

From this it follows that

$$\|\varphi_0^{-1}\overline{\Gamma}_0 - \varphi_0^{-1}\overline{\Gamma}_1\| \leqslant \|\varphi_0^{-1}\|\cdot\|\overline{K}^{-1}\|\cdot\|K\|\cdot\|\Gamma - \Gamma_n\|_\beta. \quad (12.51)$$

It may now be recalled that the mechanical quadrature formula (11.8) is accurate if it is applied to functions of the form $\Gamma_n(x)$. This means that

$$\overline{K}\varphi\Gamma_n = \varphi K\Gamma_n.$$

Hence, and from the first equality of (12.49), we have

$$\overline{K}\varphi\Gamma_n = \overline{K}\overline{\Gamma}_1.$$

It appears from the above that $\varphi\Gamma_n = \overline{\Gamma}_1$, and so, if the equality $\varphi\Gamma_n = \varphi_0\Gamma_n$ is also taken into account, we obtain

$$\Gamma_n = \varphi_0^{-1}\overline{\Gamma}_1.$$

On the basis of this equality we have

$$\Gamma - \varphi_0^{-1}\overline{\Gamma}_0 = \Gamma - \Gamma_n + \varphi_0^{-1}\overline{\Gamma}_1 - \varphi_0^{-1}\overline{\Gamma}_0.$$

Estimating the norm (i.e., the maximum of the modulus), we find

$$\|\Gamma - \varphi_0^{-1}\overline{\Gamma}_0\| \leqslant \|\Gamma - \Gamma_n\| + \|\varphi_0^{-1}\overline{\Gamma}_1 - \varphi_0^{-1}\overline{\Gamma}_0\|.$$

From this equality, by virtue of formulas (12.51), (12.48), (12.47), (12.45), (12.44) and (12.22), we obtain, finally,

$$\|\Gamma - \varphi_0^{-1}\overline{\Gamma}_0\| \leqslant \frac{A\ln n}{n^{\alpha-2\beta}}, \quad (12.52)$$

where A is a constant (depending on β and Γ).

This proves the convergence of the method since $\varphi_0^{-1}\overline{\Gamma}_0$ is nothing other than the right-hand side of formula (11.11).*

Note. As seen from the foregoing proof, if solution (12.39) is such that $\gamma(x)$ has a derivative of order $m+1$ which belongs to the class Lip α, the rapidity of convergence of the approximate solution to the exact one is determined by the inequality

$$\|\Gamma - \varphi_0^{-1}\overline{\Gamma}_0\| \leqslant \frac{A}{n^{m+\alpha-\varepsilon}}, \quad (12.53)$$

where ε is an arbitrarily small positive number and A is a constant (depending on ε and Γ).

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* In the recent paper by M. Schleiff [112] the convergence has been established for the generalized equation of wing theory [somewhat more general than Eq. (11.4)] in the space $L_2(0, \pi)$. The reader will find therein references to some of the most important works pertaining to this topic.

13. Singular Equations of the First Kind

The method of approximate solution described in the preceding section may be extended to singular equations of the form

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{\varphi(t)\,dt}{t-x}+\frac{1}{2\pi}\int_{-1}^{1}k(x,t)\,\varphi(t)\,dt=f(x), \tag{13.1}$$

where the kernel $k(x, t)$ and the free term $f(t)$ are given continuous functions of their arguments on the interval $[-1, 1]$. Equation (13.1) is frequently encountered in applications, particularly in aerodynamics and plane elasticity (see, for example, M. A. Lavrent'ev [70], H. Glauert [28], and also Chaps. III and IV of the present book).

We give a method for the approximate solution of Eq. (13.1) on the assumption that it admits an (exact) solution of one of the classes indicated below.

1°. **Solution discontinuous at the ends of the interval.** Such a solution, according to the general theory of singular integral equations (N. I. Muskhelishvili [92]), may be represented as

$$\varphi(x)=\frac{\varphi_0(x)}{\sqrt{1-x^2}}, \tag{13.2}$$

where φ_0 is a bounded function on $[-1, 1]$. We substitute expression (13.2) in Eq. (13.1) and introduce, as in Sec. 11, a new variable ϑ by the equality $x = \cos\vartheta$ $(t = \cos\tau)$, $0 \leqslant \vartheta \leqslant \pi$. Then Eq. (13.1) reduces to (retaining the previous notation for the functions φ_0, f)

$$\frac{1}{2\pi}\int_{0}^{\pi}\frac{\varphi_0(\tau)\,d\tau}{\cos\tau-\cos\vartheta}+\frac{1}{2\pi}\int_{0}^{\pi}k(\cos\vartheta,\cos\tau)\,\varphi_0(\tau)\,d\tau=f(\vartheta). \tag{13.3}$$

We construct a Lagrange interpolation polynomial for the unknown function $\varphi_0(x)$ using the Chebyshev nodes (n is a natural number):

$$x_m=\cos\vartheta_m, \qquad \vartheta_m=\frac{2m-1}{2n}\pi \qquad (m=1, \ldots, n).$$

As is known, such a polynomial has the form (see, for example, I. P. Natanson [96], p. 527)

$$L_n[\varphi_0; x]=\frac{1}{n}\sum_{\nu=1}^{n}(-1)^{\nu+1}\varphi_0(x_\nu)\frac{\cos n\vartheta\sin\vartheta_\nu}{\cos\vartheta-\cos\vartheta_\nu} \qquad (x=\cos\vartheta). \tag{13.4}$$

Note that the fraction on the right-hand side of formula (13.4) represents, for any ν, an even trigonometric polynomial of degree not greater than $n - 1$. After determining the coefficients of this polynomial by the use of the familiar equalities

$$\frac{1}{\pi}\int_0^\pi \frac{\cos n\tau\,d\tau}{\cos\tau-\cos\vartheta}=\frac{\sin n\vartheta}{\sin\vartheta} \qquad (0\leqslant\vartheta\leqslant\pi;\qquad n=0,1,\ldots), \quad (13.5)$$

we write formula (13.4) as

$$L_n[\varphi_0;x]=\frac{2}{n}\sum_{\nu=1}^{n}\varphi_\nu^0\sum_{m=0}^{n-1}\cos m\vartheta_\nu\cos m\vartheta-\frac{1}{n}\sum_{\nu=1}^{n}\varphi_\nu^0 \qquad (\varphi_\nu^0=\varphi_0(x_\nu)). \quad (13.6)$$

On the basis of the preceding two equalities we derive a quadrature formula for the improper integral:

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{\varphi(t)\,dt}{t-x}=\frac{1}{n\sin\vartheta}\sum_{\nu=1}^{n}\varphi_\nu^0\sum_{m=0}^{n-1}\cos m\vartheta_\nu\sin m\vartheta. \quad (13.7)$$

This formula is accurate whenever $\varphi_0(x)$ is a polynomial in x of degree not greater than $n-1$.

It is convenient to apply to the second integral on the left-hand side of Eq. (13.1) the Gauss-type formula (I. P. Natanson [96], p. 614):

$$\int_{-1}^{1}\frac{F(x)\,dx}{\sqrt{1-x^2}}=\frac{\pi}{n}\sum_{\nu=1}^{n}F(\cos\vartheta_\nu) \quad (13.8)$$

valid whenever $F(x)$ is a polynomial of degree not higher than $2n-1$. Based on it, we have

$$\frac{1}{2\pi}\int_{-1}^{1}k(x,t)\,\varphi(t)\,dt=\frac{1}{2n}\sum_{\nu=1}^{n}k(\cos\vartheta,\cos\vartheta_\nu)\,\varphi_0(\cos\vartheta_\nu). \quad (13.9)$$

Formulas (13.7) and (13.9) make it possible now to replace Eq. (13.1) by a system of linear algebraic equations in the approximate values of the unknown function at the nodal points. After some simplification, with the use of the equality

$$2\sum_{k=1}^{n-1}\cos k\vartheta_\nu\sin k\vartheta_m=\cot\frac{\vartheta_m\mp\vartheta_\nu}{2}$$

this system becomes

$$\sum_{\nu=1}^{n}\alpha_{m\nu}\overline{\varphi}_\nu^0=f_m \qquad (m=1,\ldots,n), \quad (13.10)$$

where

$$\alpha_{m\nu}=\frac{1}{2n}\left[\frac{1}{\sin\vartheta_m}\cot\frac{\vartheta_m\mp\vartheta_\nu}{2}+k(\cos\vartheta_m,\cos\vartheta_\nu)\right]. \quad (13.11)$$

The upper sign is taken in the case when the number $|m - \nu|$ is odd, and the lower sign when it is even.

After solving system (13.10), the approximate solution of (13.1) is found by formulas (13.2) and (13.4).

2°. **Solution bounded at one end.** In this case we assume

$$\varphi(x)=\sqrt{\frac{1-x}{1+x}}\,\varphi_0(x). \qquad (13.12)$$

The function φ_0 possesses the same properties as in representation (13.2).

Retaining the interpolation nodes unchanged, we replace φ_0 by polynomial (13.6) and substitute the result in the improper integral appearing in expression (13.1). Then, as above, a quadrature formula is obtained*:

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{\varphi(t)\,dt}{t-x}=\frac{1-\cos\vartheta}{n\sin\vartheta}\sum_{\nu=1}^{n}\varphi_\nu^0\sum_{m=0}^{n-1}\cos m\vartheta_\nu\sin m\vartheta-\frac{1}{2n}\sum_{\nu=1}^{n}\varphi_\nu^0. \qquad (13.13)$$

Formula (13.9) is replaced in this case by a similar one

$$\frac{1}{2\pi}\int_{-1}^{1}k(x,\,t)\,\varphi(t)\,dt=\frac{1}{2n}\sum_{\nu=1}^{n}(1-\cos\vartheta_\nu)\,k(\cos\vartheta,\,\cos\vartheta_\nu)\,\varphi_0(\cos\vartheta_\nu) \qquad (13.14)$$

which is accurate if the product $k(x,\,t)\,\varphi(t)$ is a polynomial in t of degree not greater than $2n-2$.

The application of the preceding formulas to Eq. (13.1) leads again to system (13.10), the coefficients $\alpha_{m\nu}$ being represented as

$$\alpha_{m\nu}=\frac{1}{2n}\Big[-1+\tan\frac{\vartheta_m}{2}\cot\frac{\vartheta_m\mp\vartheta_\nu}{2}+ +2\sin^2\frac{\vartheta_\nu}{2}\,k(\cos\vartheta_m,\,\cos\vartheta_\nu)\Big]. \qquad (13.15)$$

The signs are chosen as before.

3°. **Solution bounded at both ends.** A solution of (13.1) bounded on the interval vanishes at the ends (see N. I. Muskhelishvili [92]), $\varphi(1)=\varphi(-1)=0$, and is representable in the form

$$\varphi(x)=\sqrt{1-x^2}\,\varphi_0(x), \qquad (13.16)$$

where φ_0 is a bounded function. According to formula (13.16), Multhopp's scheme presented in Sec. 11 is extended to this case with modifications only in detail. As in Sec. 11, we shall approximate the function $\varphi(x)$ by an odd trigonometric polynomial in ϑ constructed with the use of the same nodes (11.5). This polynomial is of the form [cf. formula (11.7)]

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* A somewhat different quadrature formula for the case considered in this paragraph (when the function under the improper integral is bounded at one end only) has been derived by H. Serbin [113].

$$L_n[\varphi; x] = \frac{2}{n+1} \sum_{\nu=1}^{n} \varphi_\nu \sum_{m=1}^{n} \sin m\vartheta_\nu \sin m\vartheta. \tag{13.17}$$

From this we obtain a rather simple quadrature formula

$$\frac{1}{2\pi} \int_{-1}^{1} \frac{\varphi(t)\,dt}{t-x} = -\frac{1}{n+1} \sum_{\nu=1}^{n} \varphi_\nu \sum_{m=1}^{n} \sin m\vartheta_\nu \cos m\vartheta; \tag{13.18}$$

it is accurate whenever $\varphi(\cos\vartheta)$ is an odd trigonometric polynomial of degree not higher than n.

As before, we apply to the regular integral in Eq. (13.1) the Gauss-type formula

$$\int_{-1}^{1} \sqrt{1-x^2}\, F(x)\,dx = \frac{\pi}{n+1} \sum_{\nu=1}^{n} \sin^2 \vartheta_\nu F(\cos\vartheta_\nu)$$

which is of the same accuracy as formula (13.8). Based on it, we have (I. P. Natanson [96], p. 617)

$$\frac{1}{2\pi} \int_{-1}^{1} k(x, t)\,\varphi(t)\,dt = \frac{1}{2(n+1)} \sum_{\nu=1}^{n} \sin\vartheta_\nu k(\cos\vartheta, \cos\vartheta_\nu)\,\varphi(\cos\vartheta_\nu). \tag{13.19}$$

By using formulas (13.18) and (13.19), we again arrive at a system of the form (13.10) in φ_ν, with

$$a_{m\nu} = \frac{\sin\vartheta_\nu}{n+1} \left[\frac{\varepsilon_{m\nu}}{\cos\vartheta_\nu - \cos\vartheta_m} + \frac{1}{2} k(\cos\vartheta_m, \cos\vartheta_\nu) \right], \tag{13.20}$$

where

$$\varepsilon_{m\nu} = \begin{cases} 1 \text{ when } |m-\nu| = 1, 3, \ldots, \\ 0 \text{ when } |m-\nu| = 0, 2, \ldots . \end{cases}$$

It can readily be verified that if the functions $k(x, t)$ and $f(x)$ have x derivatives of order m which satisfy the Hölder condition on the interval $[-1, 1]$, then the function $\varphi_0(x)$ in formulas (13.2), (13.12), (13.16) has a derivative of order $m-1$ which also satisfies the Hölder condition. Because of this, the replacement of the function $\varphi_0(x)$ by an approximating polynomial, a replacement that underlies the foregoing scheme, guarantees rapid convergence of the process if the functions k and f are smooth, this being confirmed by numerous examples.

By an argument closely similar to that used in Sec. 12 it may be proved that if the function $k(x, t)$ is sufficiently smooth and Eq.

(13.1) admits a unique solution (with the required properties) for any f of a particular class, then system (13.10) is solvable for large n and the sequence of interpolation trigonometric polynomials constructed using its solution uniformly converges to the solution of (13.1).

In the practical application of the approximate method described above for the solution of (13.1) it is useful to remember the following.

The solution of class (13.2), as a rule, exists, but is not unique, whereas the solution of class (13.16) with the given right-hand side does not, in general, exist, but when it exists, it is always unique. In terms of the theory of singular equations, the index of the equation of class (13.2) is positive (equal to 1), and that of class (13.16) is negative (equal to —1). The solution of system (13.10) with matrix (13.11) for large n may, therefore, not be stable, whereas the same system with matrix (13.20), even though its solution is always stable, may not lead to the required solution.

On the other hand, singular equations that usually occur in applications originate from the properly posed physical problems and the existence of a solution belonging to one or another class of functions is guaranteed by the very nature of the problem. The solution of class (13.2) is subject to an additional condition that follows from the substance of the problem and ensures its uniqueness. In simple cases this condition is of the form

$$\int_{-1}^{1} \varphi(x)\,dx = A, \tag{13.21}$$

where A is a given constant. The preceding equation written, on the basis of (13.8), in discrete form

$$\varphi_1^0 + \varphi_2^0 + \ldots + \varphi_n^0 = A \tag{13.22}$$

is added to system (13.10) any one equation of which must be omitted. The solution of the newly obtained system of n equations will be stable.

In cases where it is necessary to have a solution bounded at both ends of the interval the right-hand side of the integral equation usually contains a constant not given beforehand. This constant is taken as the unknown, along with φ_ν, and its selection is governed by the condition for existence of a bounded solution. The latter condition (N. I. Muskhelishvili [92], Sec. 105)

$$\int_{-1}^{1} \frac{f(x) - R[\varphi;\, x]}{\sqrt{1-x^2}}\,dx = 0, \qquad R[\varphi\,;x] = \frac{1}{2\pi} \int_{-1}^{1} k(x,\, t)\,\varphi(t)\,dt \tag{13.23}$$

is taken in discrete form and added to system (13.10) with matrix (13.20). The new system of $n + 1$ equations leads to the desired result.

The procedure of solution for class (13.12) need not be modified.

We remark, finally, that a method, identical with that developed above, for the solution of (13.1) in classes (13.2) and (13.16) is contained in a recent paper by F. Erdogan and G. D. Gupta [22].

Note. In solving a singular equation by the foregoing method, it is essential that the required solutions be representable in the form

$$\varphi(x) = (1-x)^{\alpha}(1+x)^{\beta}\varphi_0(x), \qquad (13.24)$$

where $\alpha = \pm\frac{1}{2}$, $\beta = \pm\frac{1}{2}$ and $\varphi_0(x)$ is a bounded function on the interval which assumes definite values at the ends. It will be realized that, if representation (13.24) is available, the method may be extended to equations of the general form

$$a(x)\varphi(x)+\frac{b(x)}{2\pi}\int_{-1}^{1}\frac{\varphi(t)\,dt}{t-x}+\frac{1}{2\pi}\int_{-1}^{1}k(x,t)\varphi(t)\,dt=f(x). \qquad (13.25)$$

For representation (13.24) to be possible, it is sufficient, for example, that (of course, if $a^2(x)+b^2(x)\neq 0$ everywhere on $[-1, 1]$) the coefficient $a(x)$ vanish at the ends of the interval.

II. PLANE CONTACT PROBLEMS

The present part is concerned with contact problems of the plane theory of elasticity for which contact between circular bodies takes place without friction along a considerable portion of their boundaries, and consequently, for which the hypothesis of small length of contact is inapplicable. Below is given an elementary derivation of basic equations which are identical in form with the equations of the preceding two sections.

It should be noted that many of the equations that follow have been previously obtained by various authors.* The chief value of the section lies, of course, not in a new derivation of these equations, but in the application of the methods developed in Part I to an efficient solution of a class of contact problems.

14. Rigid Punch Pressed Against the Lip of a Circular Hole**

Let a rigid punch with a flat symmetrical base be pressed by a force acting along the axis of the punch into an elastic medium representing an infinite plane with a circular hole.

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* There are detailed references at appropriate places in the text.

** The present section and Secs. 15, 17, 18 reproduce the author's paper [41] with slight modifications.

It is assumed that the punch may only have a motion of translation and, further, that the stresses and rotation vanish at infinity.

The given data are taken to be the shape of the base of the punch (close to the contour of the hole) and the resultant of the external forces pressing the punch against the boundary of the medium.

We are seeking the state of stress in the elastic body.

Let the elastic medium under consideration occupy the plane of the variable* $\zeta = \xi + i\eta$ from which a circle of radius 1 with centre at the point $\zeta = 0$ is removed. Suppose that the punch is acted on by a single force of magnitude P directed oppositely to the η axis.

The boundary conditions of the problem are written in the form (cf. N. I. Muskhelishvili [93], p. 429)

$$N = 0, \quad T = 0 \text{ on } L_1, \quad T = 0, \quad v_\rho = g(\sigma) \text{ on } L_2, \qquad (14.1)$$

the last two of these conditions being fulfilled on the arc of contact L_2 which is not given beforehand, and the first two on the remainder of the complete circle L, $L = L_1 + L_2$. Here, N, T are, respectively, the normal and shearing components of the external stress acting on the contour L, v_ρ is the normal (elastic) displacement, $g(\sigma)$ is a given real function of the point $\sigma = e^{i\vartheta}$ on L_2 which characterizes the shape of the punch base and, by symmetry, satisfies the condition $g(\sigma) = g(-\overline{\sigma})$.

In the following it will be assumed that $g(\sigma)$ has a second derivative relative to the arc length of the contour which satisfies the Hölder condition.**

To solve the problem we introduce the Kolosov-Muskhelishvili potentials $\Phi(\zeta)$ and $\Psi(\zeta)$. The boundary conditions for them may be obtained from formula (1.18) (Chap. I) by setting $\omega(\zeta) = \zeta$ in it and then substituting $\zeta = \sigma$. We have***

$$\Phi(\sigma) + \overline{\Phi(\sigma)} - \sigma\Phi'(\sigma) - \sigma^2\Psi(\sigma) = N - iT \quad \text{on} \quad L. \qquad (14.2)$$

The normal stress N on the right-hand side of equality (14.2) is given only on the arc L_1 and is to be determined on the remainder of L.

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* The complex variable in a plane where the elastic medium is located is commonly denoted by z, and an auxiliary variable introduced by conformal mapping onto canonical regions by ζ. If the physical region is itself of a canonical form (bounded, for example, by a circle), we shall sometimes use the letter ζ to denote the complex variable in its plane.

** The last condition of (14.1) should be written more accurately as $v_\rho = g(\sigma) + c \sin\alpha$, where c is a translation of the punch, but this displacement may be disregarded since it can be eliminated by a rigid-body translation of the whole system.

*** By definition (see N. I. Muskhelishvili [93], pp. 146-147), N is the projection of the external stress on the normal n to the circle directed towards the centre, and T is the projection on the tangent directed to the left when viewed along n. With this choice of the signs of the external stress $\sigma_\rho = N$, $\tau_{\rho\vartheta} = T$.

It is this stress for which we shall construct a singular equation referred to as the basic equation of the problem.

According to the condition adopted above, at infinity we have on the basis of formulas (1.12), for large $|\zeta|$,

$$\Phi(\zeta)=\frac{a}{\zeta}+O(\zeta^{-2}), \qquad \Psi(\zeta)=\frac{b}{\zeta}+O(\zeta^{-2}), \tag{14.3}$$

the coefficients a, b being expressed in terms of the components of the resultant $(0, -P)$ of the external forces as follows:

$$a=\frac{iP}{2\pi(1+\varkappa)}, \qquad b=\frac{i\varkappa P}{2\pi(1+\varkappa)} \qquad \left(\varkappa=3-4\nu \text{ or } \varkappa=\frac{3-\nu}{1+\nu}\right). \tag{14.4}$$

Here ν, as always, is Poisson's ratio.

The condition $T=0$ on L is represented, by virtue of formula (14.2), in the form

$$\sigma\Phi'(\sigma)+\sigma^2\Psi(\sigma)-\bar{\sigma}\overline{\Phi'(\sigma)}-\bar{\sigma}^2\overline{\Psi(\sigma)}=0 \quad \text{on} \quad L. \tag{14.5}$$

From this, by performing the operation

$$\frac{1}{2\pi i}\int_L \frac{d\sigma}{\sigma-\zeta}$$

and taking into account formula (14.3), we immediately find

$$\zeta\Phi'(\zeta)+\zeta^2\Psi(\zeta)=b(\zeta-\zeta^{-1})+A \quad \text{for} \quad |\zeta|\geqslant 1, \tag{14.6}$$

where

$$A=\lim \zeta^2[\Psi(\zeta)-b\zeta^{-1}] \quad \text{as} \quad \zeta\to\infty.$$

To determine the constant A we multiply (14.2) by $\sigma^{-1}\,d\sigma$ and integrate the resulting equality along L. We have

$$A=-\frac{1}{2\pi i}\int_{L_2}\frac{N(\sigma)\,d\sigma}{\sigma}. \tag{14.7}$$

On the basis of (14.6) equality (14.2) becomes

$$\Phi(\sigma)+\overline{\Phi(\sigma)}-b(\sigma-\bar{\sigma})-A=N \quad \text{on} \quad L.$$

From this, as above, we find

$$\Phi(\zeta)=-\frac{1}{2\pi i}\int_{L_2}\frac{N(\sigma)\,d\sigma}{\sigma-\zeta}-\frac{b}{\zeta} \quad \text{for} \quad |\zeta|>1. \tag{14.8}$$

By differentiating expression (14.8) with respect to ζ and taking into account the continuity* of the normal stress $N(\sigma)$ on the boundary L, we obtain

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* It is assumed that the punch has no angular points on the contact boundary. Under this assumption $N(\sigma)$ is a continuous function on L which vanishes at the ends of the arc of contact.

$$\Phi'(\zeta) = -\frac{1}{2\pi i}\int_{L_2}\frac{N'(\sigma)\,d\sigma}{\sigma-\zeta} + \frac{b}{\zeta^2}. \qquad (14.9)$$

To determine $N(\sigma)$ on L_2 we make use of the formula [see formula (1.17), Chap. I]

$$2\mu v_\rho = \operatorname{Re}\{\bar{\sigma}\,[\varkappa\varphi(\sigma) - \sigma\overline{\varphi'(\sigma)} - \overline{\psi(\sigma)}]\} \quad \text{on } L. \qquad (14.10)$$

If this equality is differentiated with respect to ϑ ($\sigma = e^{i\vartheta}$) and condition (14.5) is also used, the following formula is obtained:

$$2\mu\left[v_\rho + \frac{d^2 v_\rho}{d\vartheta^2}\right] =$$
$$= \operatorname{Re}\{(\varkappa-1)\Phi(\sigma) - (\varkappa+1)\sigma\Phi'(\sigma) + \sigma\Phi'(\sigma) + \sigma^2\Psi(\sigma)\} \text{ on } L_2. \quad (14.11)$$

Inserting the limiting values of the functions $\Phi(\zeta)$, $\Phi'(\zeta)$ given by formulas (14.8), (14.9) and also the expression for the left-hand side of relation (14.6), we derive on the basis of the last of conditions (14.1) the required equation

$$\frac{\varkappa-1}{\varkappa+1}N(\sigma_0) + \frac{\sigma_0}{\pi i}\int_{L_2}\frac{N'(\sigma)\,d\sigma}{\sigma-\sigma_0} - \frac{1}{2\pi i}\int_{L_2}\frac{N(\sigma)\,d\sigma}{\sigma} + \frac{\varkappa P i}{\pi(1+\varkappa)}(\sigma_0 - \bar{\sigma}_0) =$$
$$= \frac{4\mu}{\varkappa+1}\left[g(\sigma) + \frac{d^2 g(\sigma)}{d\vartheta^2}\right] \quad \text{on } L_2. \quad (14.12)$$

For the determination of the line of integration in this singular integro-differential equation we have one more condition

$$\int_{L_2} N(\sigma)\,d\sigma = -P \qquad (14.13)$$

which expresses the specification of the resultant of the external forces applied to the lip of the hole.

After finding $N(\sigma)$ and L_2, the stress functions are determined by formulas (14.8) and (14.6).

In the special case when the rigid punch represents a circular disk inserted into a hole of the same radius, we have $g(\sigma) = 0$ on L_2 and Eq. (14.12) takes the form

$$\frac{\varkappa-1}{\varkappa+1}N(\sigma_0) + \frac{\sigma_0}{\pi i}\int_{L_2}\frac{N'(\sigma)\,d\sigma}{\sigma-\sigma_0} - \frac{1}{2\pi i}\int_{L_2}\frac{N(\sigma)\,d\sigma}{\sigma} = \frac{\varkappa P}{\pi i(1+\varkappa)}(\sigma_0 - \bar{\sigma}_0). \qquad (14.14)$$

Note. Equation (14.12) (or, more exactly, the equation obtainable from (14.14) by replacing the right-hand side by its approximate expression) is contained in the paper by V. V. Panasyuk [100]. The aforenamed author derives his equation in an entirely different way proceeding from the result of I. N. Kartsivadze.*

The case of a circular punch (both a rigid and an elastic one) inserted into a circular hole** has been considered by M. P. Sheremet'ev [114]. By a method somewhat different from the one employed by us the aforenamed author has constructed the basic equations of the problem identical with Eq. (14.14) and Eq. (15.8) of the next section in the case of generalized plane stress. In the case of plane strain, however, some (inessential) modifications should be introduced into the author's argument. By using the regularization method developed by L. G. Magnaradze [76], M. P. Sheremet'ev reduces the singular integro-differential equations of the problem to Fredholm-type integral equations. An approximate method for a direct solution of the same singular equations has been suggested by M. P. Sheremet'ev in his other paper [118].

15. Elastic Inclusion in a Medium with a Circular Hole

Let an elastic circular disk be inserted into a circular hole in an infinite elastic medium, a disk with elastic properties generally different from those of the medium and of the same radius as the hole. It is assumed that the disk is pressed against the surrounding material by a concentrated force applied at its centre. As before, we suppose that the stresses and rotation vanish at infinity.

All elements pertaining to the disk (elastic constants, stress functions, etc.) are identified by the subscript zero and the contour conditions of the problem are written in the form (cf. N. I. Muskhelishvili [93], p. 207)

$$\left.\begin{aligned} N=0,\ T=0 \text{ on } L_1,\ T=0,\ v_\rho^0=v_\rho \text{ on } L_2,\\ N_0-iT_0=N-iT \text{ on } L. \end{aligned}\right\} \qquad (15.1)$$

The stress functions $\Phi(\zeta)$, $\Psi(\zeta)$ corresponding to the infinite plate are expressed as before by formulas (14.8), (14.6).

However, the functions $\Phi_0(\zeta)$, $\Psi_0(\zeta)$ corresponding to the elastic disk, in view of the presence of the concentrated force at its centre, are of the form

$$\Phi_0(\zeta)=\frac{a_0}{\zeta}+\Phi_0^*(\zeta), \qquad \Psi_0(\zeta)=\frac{b_0}{\zeta}+\Psi_0^*(\zeta)$$
$$\left(a_0=\frac{iP}{2\pi(1+\varkappa_0)}, \quad b_0=a_0\varkappa_0\right), \qquad (15.2)$$

the functions $\Phi_0^*(\zeta)$, $\Psi_0^*(\zeta)$ being holomorphic in the unit circle.

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* An account of the results of I. N. Kartsivadze may be found, for example, in the monograph by N. I. Muskhelishvili [93] (Secs. 125, 126).

** The case of an elastic punch will be considered in the next section.

Based on the corresponding conditions (15.1), we obtain in an exactly similar way to that already described

$$\zeta\Phi_0'(\zeta)+\zeta^2\Psi_0(\zeta)=a_0(\zeta-\zeta^{-1}),\qquad |\zeta|\leqslant 1,\qquad (15.3)$$

$$\Phi_0(\zeta)=\frac{1}{2\pi i}\int\limits_{L_2}\frac{N(\sigma)\,d\sigma}{\sigma-\zeta}+a_0\left(2\zeta+\frac{1}{\zeta}\right)+\frac{A}{2},\qquad |\zeta|<1,\qquad (15.4)$$

where the constant A is given by formula (14.7); it is assumed that $\operatorname{Im}\Phi_0^*(0)=0$. From formula (15.4), after using, as before, the condition

$$N(\sigma_*)=N(-\bar{\sigma}_*)=0 \qquad (15.5)$$

(σ_*, $-\bar{\sigma}_*$ are the end points of the arc L_2), we find

$$\Phi_0'(\zeta)=\frac{1}{2\pi i}\int\limits_{L_2}\frac{N'(\sigma)\,d\sigma}{\sigma-\zeta}+a_0\left(2-\frac{1}{\zeta^2}\right). \qquad (15.6)$$

From the equality

$$v_\rho^0+\frac{d^2v_\rho^0}{d\vartheta^2}=v_\rho+\frac{d^2v_\rho}{d\vartheta^2},$$

taking formula (14.11) into account, we have

$$\mu_0\operatorname{Re}\{(\varkappa-1)\Phi(\sigma)-(\varkappa+1)\sigma\Phi'(\sigma)+\sigma\Phi'(\sigma)+\sigma^2\Psi(\sigma)\}=$$
$$=\mu\operatorname{Re}\{(\varkappa_0-1)\Phi_0(\sigma)-(\varkappa_0+1)\sigma\Phi_0'(\sigma)+\sigma\Phi_0'(\sigma)+$$
$$+\sigma^2\Psi_0(\sigma)\}\quad\text{on } L_2. \qquad (15.7)$$

Inserting the corresponding expressions given by the above formulas, we obtain, after some simplification,

$$kN(\sigma_0)+\frac{\sigma_0}{\pi i}\int\limits_{L_2}\frac{N'(\sigma)\,d\sigma}{\sigma-\sigma_0}-\frac{p}{2\pi i}\int\limits_{L_2}\frac{N(\sigma)\,d\sigma}{\sigma}=$$
$$=\frac{q}{2\pi i}P(\sigma_0-\bar{\sigma}_0)\quad\text{on } L_2, \qquad (15.8)$$

where

$$k=\frac{1}{2}\,\frac{(1-2\nu)(1+\nu)E_0-(1-2\nu_0)(1+\nu_0)E}{(1-\nu^2)E_0+(1-\nu_0^2)E}, \qquad (15.9)$$

$$p=\frac{(1-\nu^2)E_0}{(1-\nu^2)E_0+(1-\nu_0^2)E},\quad q=\frac{1}{2}\,\frac{(1+\nu)\varkappa E_0+(1+\nu_0)E}{(1-\nu^2)E_0+(1-\nu_0^2)E} \qquad (15.10)$$

and E is the modulus of elasticity. The arc of contact L_2 is still unknown, and in view of this fact to Eq. (15.8), as before, must be added relation (14.13).

In the case when the disk and plate are made of the same material ($\nu_0=\nu$, $E_0=E$), the coefficient k in formula (15.8) vanishes, and in consequence this equation is solved in closed form. The (closed) solution of the problem in this case has been found by M. Z. Narodetskii [95].

On passing in Eq. (15.8) to the limit as $E_0 \to \infty$, we come to the case of an absolutely rigid disk, namely Eq. (14.14).

When $E = \infty$, we have the case of a circular elastic disk inserted into a hole of the same radius made in an absolutely rigid plate and pressed against it by a concentrated force applied to the centre of the disk. Equation (15.8) then gives

$$-\frac{\varkappa-1}{\varkappa+1}N(\sigma_0)+\frac{\sigma_0}{\pi i}\int\limits_{L_2}\frac{N'(\sigma)\,d\sigma}{\sigma-\sigma_0}=\frac{P}{\pi i(1+\varkappa)}(\sigma_0-\bar{\sigma}_0) \text{ on } L_2, \qquad (15.11)$$

where $\varkappa$ is an elastic constant of the disk.

Equation (15.8) represents an example of the equation of a plane problem of contact between two elastic bodies treated without any restrictive assumptions. Outwardly it is of the same form as Eq. (14.12).

16. Region of Contact Given

We shall take up the problem of Sec. 14, but reject the assumption of smoothness of the base of a rigid punch which is flat and symmetrical as before. More precisely, we shall consider the problem of the equilibrium of a rigid punch on the lip of a circular hole when the region of contact of the punch with the elastic plate, i.e., the contact line L_2 is given beforehand. Contact with a given segment of contact may take place in two cases: either when the end points of the punch base lie on the lip of the hole before deformation and the punch base comes entirely in contact with the elastic body or when the punch base has angular points originally not located on the lip of the hole but making contact with the body due to the action of the pressing force P (see N. I. Muskhelishvili [93], Sec. 116).

In the derivation of the integro-differential equation of the contact problem considered in Sec. 14 [Eq. (14.12)] of basic importance was the assumption of continuity of the normal stress $N(\sigma)$ at the ends of the line L_2. In our case, however, when L_2 is given, this assumption ceases to be valid: the function $N(\sigma)$, being continuous on the open arcs L_1 and L_2, have singularities of lower order than unity at the ends of these arcs. For this reason the argument used in Sec. 14 is to be essentially modified in our case.

In all other respects we retain the assumptions of Sec. 14. In particular, the base of the punch is assumed to be symmetrical, i.e., the arc L_2 is symmetrical about the imaginary axis, and the given function $g(\sigma)$ is also symmetrical: $g(\sigma) = g(-\bar{\sigma})$ on L_2. The region $|\zeta| > 1$ occupied by the elastic medium will be denoted by Σ^-, and the inside of the unit circle $|\zeta| < 1$ by Σ^+.

Proceeding to the solution of the problem, we write the first pair of conditions (14.1), expressing the absence of external forces on L_1, in the form

$$\varphi(\sigma) + \overline{\sigma\varphi'(\sigma)} + \overline{\psi(\sigma)} = \text{constant on } L_1, \qquad (16.1)$$

where $\varphi(\zeta)$, $\psi(\zeta)$ are the unknown analytic functions in the region Σ^- which have the form*

$$\varphi(\zeta) = a \ln \zeta + \varphi_0(\zeta), \qquad \psi(\zeta) = b \ln \zeta + \psi_0(\zeta), \qquad (16.2)$$

where $\varphi_0(\zeta)$, $\psi_0(\zeta)$ are holomorphic functions in Σ^- and the constants a and b are given by equalities (14.4).

In terms of the functions Φ and Ψ condition (16.1) would be of the form

$$\Phi(\sigma) + \overline{\Phi(\sigma)} - \sigma\Phi'(\sigma) - \sigma^2\Psi(\sigma) = 0 \text{ on } L_1; \qquad (16.3)$$

it will be recalled that

$$\varphi'(\zeta) = \Phi(\zeta), \qquad \psi'(\zeta) = \Psi(\zeta).$$

Condition (16.1) compares favourably with condition (16.3) in the sense that it does not contain boundary values of the unknown potentials φ and ψ of higher orders. However, condition (16.1) in turn involves some inconvenience, namely it includes boundary values of multiple-valued functions.

In view of the above we have to express the second pair of the boundary conditions (14.1) in a form similar to condition (16.1). For this purpose, we use the well-known formula

$$T = \tau_{\rho\vartheta} = \operatorname{Im}\{[\overline{\sigma\varphi''(\sigma)} + \psi'(\sigma)]\,\sigma^2\} \qquad (\sigma = e^{i\vartheta}),$$

add to it formula (14.10) and note also that the boundary of the region represents a circle of unit radius. After elementary computation we obtain the following relation:

$$2\mu v_\rho + i\left(2\mu\frac{dv_\rho}{d\vartheta} - \tau_{\rho\vartheta}\right) =$$

$$= \bar{\sigma}\,[\varkappa\varphi(\sigma) - \overline{\sigma\varphi'(\sigma)} - \overline{\psi(\sigma)}] - i(\varkappa + 1)\operatorname{Im}\varphi'(\sigma).$$

From this it follows that the condition for rigid contact with an elastic plate can be represented through a single complex-valued equality in the form

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* In contrast to Sec. 14, we give up using the functions $\Phi(\zeta)$ and $\Psi(\zeta)$ in the boundary conditions. The point is that if the functions $\Phi(\zeta)$ and $\Psi(\zeta)$ are introduced according to the conditions of the problem (14.1) we cannot avoid considering the function $\Phi'(\zeta)$ which in our case is not sectionally holomorphic in the plane cut along the line L_2 (for the definition of a sectionally holomorphic function see, for example, N. I. Muskhelishvili [93], Sec. 106). Consequently, it will be impossible to represent this function in the form of a Cauchy-type integral with a density function from the class H^*, and the reduction of the problem to an integral equation of the type studied will be difficult.

$$\bar{\sigma}\,[\varkappa\varphi(\sigma) - \overline{\sigma\varphi'(\sigma)} - \overline{\psi(\sigma)}] - (\varkappa + 1)\,\mathrm{Im}\,\varphi'(\sigma) = g(\sigma) \quad \text{on } L_2, \tag{16.4}$$

where

$$g(\sigma) = 2\mu v_\rho + i\left(2\mu\frac{dv_\rho}{d\vartheta} - \tau_{\rho\vartheta}\right).$$

Equalities (16.1) and (16.4) express the boundary conditions of our contact problem.

The definition of the complex potential $\varphi(\zeta)$ may be extended to the region Σ^+ by the equality*

$$\varphi(\zeta) = -\zeta\bar{\varphi}'\left(\frac{1}{\zeta}\right) - \bar{\psi}\left(\frac{1}{\zeta}\right) + \text{constant} \quad (\zeta \text{ in } \Sigma^+), \tag{16.5}$$

where use has been made of the notation (N. I. Muskhelishvili [93], Sec. 76)

$$\bar{F}\left(\frac{1}{\zeta}\right) = \overline{F\left(\frac{1}{\bar{\zeta}}\right)}. \tag{16.6}$$

The function $\varphi(\zeta)$ defined by formula (16.5) is analytic in Σ^+, with the exception of the point $\zeta = 0$, where it has a logarithmic singularity. More comprehensively, on the basis of representations (16.2) and definition (16.6) we have from formula (16.5)

$$\varphi(\zeta) = -b\ln\zeta + \text{holomorphic function} \quad (\zeta \text{ in } \Sigma^+). \tag{16.7}$$

Thus, the function $\varphi(\zeta)$ is defined in both the region Σ^- and the region Σ^+ and is of the form

$$\left.\begin{aligned} \varphi(\zeta) &= a\ln\zeta + \text{holomorphic function} \quad (\zeta \text{ in } \Sigma^-), \\ \varphi(\zeta) &= -b\ln\zeta + \text{holomorphic function} \quad (\zeta \text{ in } \Sigma^+). \end{aligned}\right\} \tag{16.8}$$

We now replace the point ζ by $\frac{1}{\bar{\zeta}}$ in equality (16.5), assuming that $|\zeta| > 1$, and take the conjugate of this equality. As a result we have

$$\bar{\varphi}\left(\frac{1}{\zeta}\right) = -\frac{1}{\zeta}\varphi'(\zeta) - \psi(\zeta) + \text{constant} \quad (\zeta \text{ in } \Sigma^-). \tag{16.9}$$

From this the unknown complex potential $\psi(\zeta)$ is determined in terms of $\varphi(\zeta)$ as

$$\psi(\zeta) = -\bar{\varphi}\left(\frac{1}{\zeta}\right) - \frac{1}{\zeta}\varphi'(\zeta) + \text{constant} \quad (\zeta \text{ in } \Sigma^-). \tag{16.10}$$

Based on formula (16.8), it is easily verified that representation (16.2) holds true for the function $\psi(\zeta)$.

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* Formula (16.5) is obtained by integration with respect to ζ of the formula of I. N. Kartsivadze [see, for example, N. I. Muskhelishvili [93], Sec. 121, formula (9)].

By allowing the point ζ to approach a point $\sigma = e^{i\vartheta}$ from the region Σ^-, we obtain

$$\psi(\sigma) = -\overline{\varphi}^+(\overline{\sigma}) - \overline{\sigma}\varphi'^-(\sigma) + \text{constant}. \tag{16.11}$$

The boundary values $\psi(\sigma)$ from formula (16.11) are inserted in the conditions of the problem (16.1) and (16.4). Accordingly we have

$$\varphi^-(\sigma) - \varphi^+(\sigma) = \text{constant on } L_1, \tag{16.12}$$

$$\overline{\sigma}\,[\varkappa\varphi^-(\sigma) + \varphi^+(\sigma) + \text{constant}] - i(\varkappa + 1)\,\mathrm{Im}\,\varphi'^-(\sigma) = g(\sigma) \quad \text{on } L_2. \tag{16.13}$$

According to the condition of our problem following from the general requirements of classical elasticity, the normal stress $\sigma_\rho = N$ may have only an integrable singularity at the ends of the line L_2. Under this condition, of course, formula (14.8) also holds in our case. Thus,

$$\varphi'(\zeta) = -\frac{1}{2\pi i}\int_{L_2}\frac{N(\sigma)\,d\sigma}{\sigma-\zeta} - \frac{b}{\zeta}. \tag{16.14}$$

In the above formula the point ζ may be taken to be any point in the plane, with the exception of $\zeta = 0$ and points located on L_2. In other words, the function $\varphi'(\zeta)$ is sectionally holomorphic in the plane with the boundary L_2 and has a pole of order unity at the origin. By integrating relation (16.14), we find

$$\varphi(\zeta) = \frac{1}{2\pi i}\int_{L_2} N(\sigma)\ln\left(1 - \frac{\zeta}{\sigma}\right)d\sigma - b\ln\zeta + \text{constant}. \tag{16.15}$$

This function will be considered in both the region Σ^+ and the region Σ^- but the points $\zeta = 0, \infty$ will be excluded. By $\ln\left(1 - \frac{\zeta}{\sigma}\right)$ for fixed σ is meant a holomorphic function in the region Σ^+ which vanishes at $\zeta = 0$. For $|\zeta| > 1$ we can write (σ is an arbitrary point of the arc L_2)

$$\ln\left(1 - \frac{\zeta}{\sigma}\right) = \ln(\sigma - \zeta) - \ln\sigma = \ln\zeta + \ln\left(\frac{\sigma}{\zeta} - 1\right) - \ln\sigma =$$

$$= \ln\zeta + \ln\left(1 - \frac{\sigma}{\zeta}\right) - \ln\sigma + (2k+1)\pi i \quad (k \text{ an integer}). \tag{16.16}$$

The function $\ln\left(1 - \frac{\sigma}{\zeta}\right)$ is, of course, holomorphic in ζ in the region Σ^-. Consequently, in this region $\ln\left(1 - \frac{\zeta}{\sigma}\right)$ is made up of a holomorphic function and a function of the type $\ln\zeta$.

On the assumption that ζ belongs to the region Σ^- let us evaluate the coefficient of $\ln\zeta$ on the right-hand side of relation (16.15).

According to formulas (14.13) and (16.16) this coefficient is equal to

$$-b+\frac{1}{2\pi i}\int\limits_{L_2} N(\sigma)\,d\sigma=-b+\frac{Pi}{2\pi}=\frac{Pi}{2\pi(1+\varkappa)}=a.$$

It appears from the above that the function $\varphi(\zeta)$ is representable in the form*

$$\left.\begin{aligned}\varphi(\zeta)&=\frac{1}{2\pi i}\int\limits_{L_2} N(\sigma)\ln\left(1-\frac{\sigma}{\zeta}\right)d\sigma+a\ln\zeta+C^-\ (\zeta \text{ in } \Sigma^-),\\ \varphi(\zeta)&=\frac{1}{2\pi i}\int\limits_{L_2} N(\sigma)\ln\left(1-\frac{\zeta}{\sigma}\right)d\sigma-b\ln\zeta+C^+\ (\zeta \text{ in } \Sigma^+),\end{aligned}\right\}\quad(16.17)$$

where C^- and C^+ are arbitrary complex constants.

From formula (16.17) it follows immediately that conditions (16.8) are satisfied.

Let us now find the limiting values of the function $\varphi(\zeta)$ on L ($L = L_1 + L_2$). From formulas (16.15) or (16.17) we have

$$\left.\begin{aligned}\varphi^-(\sigma_0)&=\frac{1}{2\pi i}\int\limits_{L_2} N(\sigma)\ln\left(1-\frac{\sigma_0}{\sigma}\right)d\sigma-b\ln\sigma_0+C^-,\\ \varphi^+(\sigma_0)&=\frac{1}{2\pi i}\int\limits_{L_2} N(\sigma)\ln\left(1-\frac{\sigma_0}{\sigma}\right)d\sigma-b\ln\sigma_0+C^+\ (\text{on } L),\end{aligned}\right\}\quad(16.18)$$

where by $\ln\left(1-\frac{\sigma_0}{\sigma}\right)$ are meant the limiting values of the holomorphic function $\ln\left(1-\frac{\zeta}{\sigma}\right)$ in the region Σ^+ when the point ζ tends to a point σ_0 on L, remaining within Σ^+ at all times.

On the basis of the above formulas condition (16.12) is fulfilled.

We now turn to equality (16.13) and note that the condition $\tau_{\rho\vartheta} = 0$ on L has already been used in the derivation of the integral representation (16.14). If we also recall the expression for $g(\sigma)$, it becomes quite evident that we may restrict our consideration to a real or an imaginary part of equality (16.13). The imaginary part of the latter gives

$$\operatorname{Im}\{\bar{\sigma}[\varkappa\varphi^-(\sigma)+\varphi^+(\sigma)]+\text{constant}\}-(\varkappa+1)\operatorname{Im}\varphi'^-(\sigma)=$$
$$=g_0(\sigma)\quad\text{on}\quad L_2\qquad(16.19)$$

with

$$g_0(\sigma)=2\mu\frac{dv_\rho}{d\vartheta}.\qquad(16.20)$$

For the limiting values of φ^- and φ^+ in equality (16.19) we make use of formulas (16.18) and for the values of φ'^- we employ the

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* Holomorphic functions of the type of integrals on the right-hand sides of formulas (16.17) have been used in early investigations of I. N. Vekua [137].

Sokhotskii-Plemelj formula

$$\varphi'^{-}(\sigma_0)=\frac{N(\sigma_0)}{2}-\frac{1}{2\pi i}\int\limits_{L_2}\frac{N(\sigma)\,d\sigma}{\sigma-\sigma_0}-\frac{b}{\sigma_0}\quad(\text{on } L_2). \tag{16.21}$$

We have ($\sigma=e^{i\vartheta}$)

$$\frac{d\sigma}{\sigma-\sigma_0}=\frac{1}{2}\cot\frac{\vartheta-\vartheta_0}{2}\,d\vartheta+\frac{i\,d\vartheta}{2},\qquad d\vartheta=\frac{d\sigma}{i\sigma}.$$

Hence,

$$\operatorname{Im}\left\{\frac{i\,d\sigma}{\sigma-\sigma_0}\right\}=\frac{1}{2}\cot\frac{\vartheta-\vartheta_0}{2}\,d\vartheta=\frac{d\sigma}{\sigma-\sigma_0}-\frac{d\sigma}{2\sigma}.$$

Consequently,

$$\operatorname{Im}\varphi'^{-}(\sigma_0)=\frac{1}{2\pi}\int\limits_{L_2}\left[\frac{1}{\sigma-\sigma_0}-\frac{1}{2\sigma}\right]N(\sigma)\,d\sigma-\frac{b}{2i}(\sigma_0+\bar{\sigma}_0). \tag{16.22}$$

Substituting the above values in formula (16.19), we find

$$-\frac{1}{2\pi}\int\limits_{L_2}\left[\frac{1}{\sigma-\sigma_0}-\frac{1}{2\sigma}\right]N(\sigma)\,d\sigma+$$

$$+\frac{1}{2\pi i}\int\limits_{L_2}\operatorname{Im}\left\{\frac{\sigma}{\sigma_0}\ln\left(1-\frac{\sigma_0}{\sigma}\right)\frac{N(\sigma)\,d\sigma}{\sigma}\right\}+\operatorname{Im}\{C\bar{\sigma}_0\}=$$

$$=\frac{bi}{2}[\sigma_0+\bar{\sigma}_0+(\sigma_0-\bar{\sigma}_0)\ln\sigma_0]+f_0(\sigma_0)\ \text{ on } L_2, \tag{16.23}$$

$$f_0(\sigma)=\frac{2\mu}{\varkappa+1}\frac{dv_\rho}{d\vartheta}, \tag{16.24}$$

where C is an arbitrary complex constant and $\ln\sigma$ on L_2 is understood to be the principal value of the logarithm.

Assuming $C=C_1+iC_2$, we have

$$\operatorname{Im}\{C\bar{\sigma}_0\}=-C_1\sin\vartheta_0+C_2\cos\vartheta_0.$$

From the meaning of the function $f_0(\sigma_0)$ defined by equality (16.24) it is obvious that a term of the form of $C_2\cos\vartheta$ can always be neglected since it expresses a rigid-body displacement of the whole system consisting of the punch and the elastic plate.* So far as the constant C_1 is concerned, it can be determined in the form of some functional of the unknown $N(\sigma)$ [when the function $f(\sigma)$ is given] from Eq. (16.23) itself. By setting $\sigma_0 = -i$ $\left(\vartheta_0 = \frac{3\pi}{2}\right)$, we find*

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* See the second footnote on p. 118.

$$C_1 = f_0(-i) - \frac{3\pi}{2} b_0 + \frac{1}{2\pi} \int\limits_{L_2} \left[\frac{\sigma - i}{2\sigma(\sigma + i)}\right] N(\sigma)\, d\sigma -$$
$$- \frac{1}{2\pi i} \int\limits_{L_2} \operatorname{Im}\left\{ i\sigma \ln\left(1 - \frac{1}{i\sigma}\right)\right\} \frac{N(\sigma)\, d\sigma}{\sigma},$$
$$b = ib_0. \tag{16.25}$$

After that Eq. (16.23) takes the form

$$-\frac{1}{2\pi} \int\limits_{L_2} \left[\frac{1}{\sigma - \sigma_0} - \frac{1}{2\sigma}\right] N(\sigma)\, d\sigma +$$
$$+ \frac{1}{2\pi i} \int\limits_{L_2} \operatorname{Im}\left\{\frac{\sigma}{\sigma_0} \ln\left(1 - \frac{\sigma_0}{\sigma}\right)\right\} \frac{N(\sigma)\, d\sigma}{\sigma} - \frac{C_1}{2i}(\sigma_0 - \bar{\sigma}_0) =$$
$$= \frac{bi}{2}[\sigma_0 + \bar{\sigma}_0 + (\sigma_0 - \bar{\sigma}_0) \ln \sigma_0] + f_0(\sigma_0) \quad \text{on } L_2, \tag{16.26}$$

where the real constant C_1 is defined by formula (16.25).

This is the integral equation of our problem, a real singular integral equation for the determination of the (real) unknown pressure under the punch $N(\sigma)$. This equation is of the form

$$\frac{1}{2\pi} \int\limits_{L_2} \frac{N(\sigma)\, d\sigma}{\sigma - \sigma_0} + \frac{1}{2\pi} \int\limits_{L_2} K(\sigma_0, \sigma) N(\sigma)\, d\sigma = F(\sigma_0) \quad \text{on } L_2,$$

where the kernel $K(\sigma_0, \sigma)$ is quasi-regular. The function $K(\sigma_0, \sigma)$ has a weak singularity at $\sigma = \sigma_0$ of the type of $\ln|\sigma - \sigma_0|$ and a fixed singularity of order unity at $\sigma = -i$. In the next section we shall reduce Eq. (16.26) to a somewhat different form convenient for computational purposes.

In the special case when the rigid punch represents a circular arc of unit radius (symmetrical about the imaginary axis), we have

$$v_\rho(\sigma) = \text{constant}, \quad f_0(\sigma) = 0 \quad \text{on} \quad L_2$$

and there will be Eq. (16.26) left without the function $f_0(\sigma_0)$.

To Eq. (16.26) must, of course, be added condition (14.13) which expresses the existence of the resultant of the external forces applied through the punch to the boundary of the elastic body.

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* According to the formulation of the problem, the point $\sigma = -i$ is always located on the contact line.

17. Solution of the Equations of the Contact Problem

We insert in Eq. (15.8) a new variable x introduced by the relation

$$\sigma = i\frac{x-i\beta}{x+i\beta}, \qquad \beta = \frac{\cos\vartheta_*}{1+\sin\vartheta_*} \qquad (\sigma_* = e^{i\vartheta_*}). \tag{17.1}$$

As is known, the above function gives a transformation which maps the unit circle $|\sigma| = 1$ onto the real x axis, the arc L_2 being transformed into the interval $[-1, 1]$; the points of the circle $-\sigma_*$, σ_* will map* into the points $x = -1$, $x = 1$, respectively.

After obvious transformations we have

$$\frac{k\beta}{x^2+\beta^2}N(x) - \frac{1}{2\pi}\int_{-1}^{1}\frac{N'(t)\,dt}{t-x} - \frac{p\beta^2}{\pi(x^2+\beta^2)}\int_{-1}^{1}\frac{N(t)\,dt}{t^2+\beta^2} = \frac{q\beta}{\pi}P\frac{x^2-\beta^2}{(x^2+\beta^2)^2} \tag{17.2}$$

with

$$N(1) = N(-1) = 0$$

(for simplicity we again assume that $N(\sigma) = N(x)$). Equality (14.13) transforms to

$$2\beta\int_{-1}^{1}\frac{t^2-\beta^2}{(t^2+\beta^2)^2}N(t)\,dt = P. \tag{17.3}$$

Equation (17.2) contains the unknown arc of contact through the parameter β which is to be determined (together with $N(x)$) from the additional condition (17.3).

This equation is identical in form, apart from the unknown parameter β and the functional of the simplest kind on its left-hand side, with the equation of wing theory (11.1), and the method of approximate solution presented in Sec. 11 applies to it.

Solving Eqs. (17.2) and (17.3) simultaneously by the above method gives an approximate solution of (17.2) in the form [see formula (11.7)]

$$N(x) = \frac{2\sqrt{1-x^2}}{n+1}\sum_{m=1}^{n} a_m U_{m-1}(x), \tag{17.4}$$

where

$$a_m = \sum_{k=1}^{n} \bar{N}_k \sin m\vartheta_k, \qquad U_{m-1}(x) = \frac{\sin m\vartheta}{\sin\vartheta}.$$

The function $U_{m-1}(x)$ represents a polynomial of degree $m-1$ in x, viz. a Chebyshev polynomial of the second kind.

---

* σ_* denotes the terminal point of the arc L_2 for which $\operatorname{Re}\sigma_* > 0$.

For the calculation of the unknown quantities of the problem it is onvenient to transform from the physical plane of the variable ζ o the plane of the variable $z = x + iy$ by means of the relation

$$\zeta = i\frac{z-i\beta}{z+i\beta} \qquad \left(\beta = \frac{\cos\vartheta_*}{1+\sin\vartheta_*}\right) \tag{17.5}$$

hich effects a conformal mapping of the circle $|\zeta| < 1$ onto the pper half-plane $\operatorname{Im} z > 0$. After some manipulation we find the ormula

$$\frac{1}{2\pi i}\int_{L_2}\frac{N(\sigma)\,d\sigma}{\sigma-\zeta} = \frac{1}{2\pi i}\int_{-1}^{1}\frac{N(x)\,dx}{x-z} + \frac{\beta}{2\pi}\int_{-1}^{1}\frac{N(x)\,dx}{x^2+\beta^2}. \tag{17.6}$$

But the Cauchy-type integral on the right-hand side of formula 17.6) is evaluated in finite form if $N(x)$ is of the form given by 17.4). Consequently, after solving the equation of the contact roblem by Multhopp's method, the corresponding stress functions re determined in closed form. Detailed calculations will be given n the next section for specific numerical examples.

So far we have only examined the equations of the contact problem hen the line of contact between two bodies is not given beforehand. Ve now turn to Eq. (16.26) with a prescribed interval of integraion L_2.

Here, too, we use substitution (17.1) and note that the parameter in our case is completely determined by the equality

$$\beta = \frac{\cos\vartheta_*}{1+\sin\vartheta_*},$$

here ϑ_* is the polar angle of the right end of the line L_2.

We present some elementary formulas relating to substitution 17.1) and necessary for the transformation of Eq. (16.26)

$$\frac{d\sigma}{\sigma-\sigma_0} = \frac{x_0+i\beta}{x+i\beta}\,\frac{dx}{x-x_0}, \qquad \frac{d\sigma}{2\sigma} = i\beta\,\frac{dx}{x^2+\beta^2},$$

$$\left(\frac{1}{\sigma-\sigma_0} - \frac{1}{2\sigma}\right)d\sigma = \left(\frac{1}{x-x_0} - \frac{x}{x^2+\beta^2}\right)dx,$$

$$\frac{\sigma+\bar\sigma}{2} = \frac{2\beta x}{x^2+\beta^2}, \qquad \frac{\sigma-\bar\sigma}{2} = i\,\frac{x^2-\beta^2}{x^2+\beta^2},$$

$$\mathrm{m}\left\{\frac{\sigma}{\sigma_0}\ln\left(1-\frac{\sigma_0}{\sigma}\right)\right\}\frac{d\sigma}{i\sigma} = X(x_0, x)\arctan Q(x_0, x) + \\ + Y(x_0, x)\ln R(x_0, x),$$

here

$$\left.\begin{aligned} X(x_0, x) &= 2\beta\,\frac{(x^2-\beta^2)(x_0^2-\beta^2)+4\beta^2 x x_0}{(x^2+\beta^2)^2(x_0^2+\beta^2)}, \\ Q(x_0, x) &= \frac{x^2+\beta^2}{\beta(x-x_0)} - \frac{x}{\beta}, \end{aligned}\right\} \tag{17.7}$$

and

$$\left.\begin{aligned} Y(x_0, x) &= 4\beta^2\frac{x_0(x^2-\beta^2)-x(x_0^2-\beta^2)}{(x^2+\beta^2)^2(x_0^2+\beta^2)}, \\ R(x_0, x) &= \frac{2\beta\,|x-x_0|}{\sqrt{(x^2+\beta^2)^2(x_0^2+\beta^2)}}. \end{aligned}\right\} \tag{17.8}$$

The above formulas are inserted in Eq. (16.26), retaining the same notation for the functions $N(\sigma)$ and $f_0(\sigma)$ and replacing x_0 by x and x by t. The transformed equation (16.26), if formula (16.25) is taken into account, is then

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{N(t)\,dt}{t-x}+\frac{1}{2\pi}\int_{-1}^{1}k(x, t)\,N(t)\,dt=f(x) \quad \text{on } (-1, 1), \tag{17.9}$$

$$\begin{aligned} k(x, t) = &-\left[\frac{t}{t^2+\beta^2}+X(x, t)\arctan Q(x, t)+Y(x, t)\ln R(x, t)\right]+ \\ &+\frac{x^2-\beta^2}{x^2+\beta^2}\left[\frac{\beta^2}{t(t^2+\beta^2)}-X(0, t)\arctan Q(0, t)-Y(0, t)\ln R(0, t)\right], \end{aligned} \tag{17.10}$$

$$\begin{aligned} f(x) = &-f_0(x)+\frac{x^2-\beta^2}{x^2+\beta^2}\left[\frac{3\pi b_0}{2}-f_0(0)\right]+ \\ &+\frac{b_0}{x^2+\beta^2}\left[2\beta x-(x^2+\beta^2)\arctan\frac{x^2-\beta^2}{2\beta x}\right]. \end{aligned} \tag{17.11}$$

The functions $X(x, t)$, $Y(x, t)$, $Q(x, t)$, $R(x, t)$ are defined by formulas (17.7) and (17.8).

As seen from formula (17.10), the kernel $k(x, t)$ possesses a weak singularity of the type $\ln|x-t|$ at $x=t$ and a singularity of order unity at $t=0$.* Nevertheless, the method of approximate solution for the equation of the first kind presented in Sec. 13 is applicable with slight modifications to Eq. (17.9). According to the physical significance of the problem our interest will be centred on solutions of (17.9) that are not bounded at the ends of the interval (Sec. 13, Par. 1°).

18. Examples

1. Rigid circular disk pressed against a hole in an elastic plate (exterior problem). The corresponding equation is obtained from Eq. (14.14) by substituting expression (17.1) or by a limiting process in formula (17.2) as $E_0 \to \infty$. This equation is written as

$$\frac{\varkappa-1}{\varkappa+1}\,\frac{\beta}{x^2+\beta^2}\,N(x)-\frac{1}{2\pi}\int_{-1}^{1}\frac{N'(t)\,dt}{t-x}=\frac{\beta^2 q}{x^2+\beta^2}+\frac{2\varkappa\beta P}{\pi(1+\varkappa)}\,\frac{x^2-\beta^2}{(x^2+\beta^2)^2}, \tag{18.1}$$

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* Under transformation (17.1) the point of the circle $\sigma=-i$ maps into the point $x=0$ of the real axis. This fixed singularity could be omitted altogether in our case since, in view of the symmetry of the problem, the functionals containing it vanish.

where

$$q=\frac{1}{\pi}\int_{-1}^{1}\frac{N(t)\,dt}{t^2+\beta^2}. \tag{18.2}$$

It may be recalled that the parameter β appearing in Eq. (18.1) and characterizing the length of the arc of contact is not known and, in addition, the equation contains on the right-hand side the constant q in the form of functional (18.2), not yet determined. In other respects Eq. (18.1) is identical with Eq. (11.1); here $B(x) > 0$ on $[-1, 1]$.

Equality (17.3) is rewritten as

$$c(\beta)\equiv\frac{2\beta}{P}\int_{-1}^{1}\frac{t^2-\beta^2}{(t^2+\beta^2)^2}N(t)\,dt=1. \tag{18.3}$$

In our (symmetrical) case of Eq. (18.1) system (11.9) for odd n is of the form

$$\left[\frac{\varkappa-1}{\varkappa+1}\frac{\beta}{\cos^2\vartheta_m+\beta^2}+b_{mm}\right]\overline{N}_m=\sum_{k=1}^{1/2(n+1)}B_{mk}\overline{N}_k+\frac{\beta^2 q}{\cos^2\vartheta_m+\beta^2}+$$

$$+\frac{2\varkappa P\beta}{\pi(1+\varkappa)}\frac{\cos^2\vartheta_m-\beta^2}{(\cos^2\vartheta_m+\beta^2)^2}, \tag{18.4}$$

where

$$\overline{\Gamma}_k=\overline{\Gamma}_{n+1-k}\quad\left(m=1,2,\ldots,\frac{1}{2}(n+1)\right),\quad B_{mk}=b_{mk}+$$

$$+b_{m,\,n+1-k}\quad\left(k=1,\ldots,\frac{1}{2}(n-1)\right),\quad B_{m,\frac{1}{2}(n+1)}=b_{m,\frac{1}{2}(n+1)} \tag{18.5}$$

and the coefficients b_{mk} are defined by formulas (11.10).

By solving this system for a certain specific value of the parameter β (say, $\beta = 1$), we find the unknowns $\overline{N}_k$ as

$$\overline{N}_k=N_k^{(0)}q+N_k^{(1)}\qquad(k=1,\ldots,n), \tag{18.6}$$

$N_k^{(0)}$, $N_k^{(1)}$ being known numbers. The corresponding approximate solution $\overline{N}(x)$ will obviously contain the constant q to be determined subsequently from formula (18.2). After determining $\overline{N}(x)$, the left-hand side of equality (18.3) is calculated next. For the determination of q and $c(\beta)$ use is made of the Gauss-type quadrature formula (see, for example, I. P. Natanson [96], p. 617)

$$\left.\begin{aligned} q &= \frac{1}{n+1}\sum_{k=1}^{n} \frac{\sin\vartheta_k \overline{N}_k}{\cos^2\vartheta_k+\beta^2}, \\ c(\beta) &\equiv \frac{2\beta\pi}{n+1}\sum_{k+1}^{n} \frac{\sin\vartheta_k(\cos^2\vartheta_k-\beta^2)}{(\cos^2\vartheta_k+\beta^2)^2}\frac{\overline{N}_k}{P}=1. \end{aligned}\right\} \tag{18.7}$$

The value of β taken by us and the corresponding $N(x)$ will not, in general, satisfy equality (18.3). By selecting new values of β, we shall repeat our calculations, therefore, until the above equality is satisfied with the required accuracy. As a result we shall have some approximate values of β and $N(x)$ for a given n. Next, other values of n are taken and the calculations indicated are repeated.

To begin with, we set $n = 7$ in system (18.4) and take Poisson's ratio equal to 1/3 for subsequent calculations. The resulting system in expanded form is represented as*

$$\left[\frac{\beta}{4(\cos^2\vartheta_1+\beta^2)}+5.2262\right]\overline{N}_1=1.9142\overline{N}_2+0.1464\overline{N}_4+ \\ +\frac{q\beta^2}{\cos^2\vartheta_1+\beta^2}+\frac{5}{4}\beta\frac{\cos^2\vartheta_1-\beta^2}{(\cos^2\vartheta_1+\beta^2)^2}\frac{P}{\pi}, \tag{18.8a}$$

$$\left[\frac{\beta}{4(\cos^2\vartheta_2+\beta^2)}+2.8284\right]\overline{N}_2=1.0360\overline{N}_1+1.1944\overline{N}_3+ \\ +\frac{q\beta^2}{\cos^2\vartheta_2+\beta^2}+\frac{5}{4}\beta\frac{\cos^2\vartheta_2-\beta^2}{(\cos^2\vartheta_2+\beta^2)^2}\frac{P}{\pi}, \tag{18.8b}$$

$$\left[\frac{\beta}{4(\cos^2\vartheta_3+\beta^2)}+2.1648\right]\overline{N}_3=0.9142\overline{N}_2+0.8536\overline{N}_4+ \\ +\frac{q\beta^2}{\cos^2\vartheta_3+\beta^2}+\frac{5}{4}\beta\frac{\cos^2\vartheta_3-\beta^2}{(\cos^2\vartheta_3+\beta^2)^2}\frac{P}{\pi}, \tag{18.8c}$$

$$\left[\frac{\beta}{4(\cos^2\vartheta_4+\beta^2)}+2.0000\right]\overline{N}_4=0.1121\overline{N}_1+1.5774\overline{N}_3+ \\ +\frac{q\beta^2}{\cos^2\vartheta_4+\beta^2}+\frac{5}{4}\beta\frac{\cos^2\vartheta_4-\beta^2}{(\cos^2\vartheta_4+\beta^2)^2}\frac{P}{\pi}, \tag{18.8d}$$

where

$$\vartheta_k=\frac{k\pi}{8} \qquad (k=1, 2, 3, 4).$$

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* In the above-cited paper of H. Multhopp [87] there are tables for the numerical values of the coefficients b_{mk} for $n = 7, 15, 31$.

By solving this system and relation (18.7) simultaneously, we have [the values for q and $N_k^{(1)}$ in formula (18.6) and also $\overline{N}_k$ are given in terms of P/π]*

$$\beta = 1.20886, \qquad q = -0.52422, \tag{18.9}$$

$$\left.\begin{aligned} \overline{N}_1 &= 0.48035q - 0.29067 = -0.54248, \\ \overline{N}_2 &= 0.91050q - 0.63176 = -1.10906, \\ \overline{N}_3 &= 1.23317q - 1.00926 = -1.65571, \\ \overline{N}_4 &= 1.35900q - 1.20474 = -1.91715. \end{aligned}\right\} \tag{18.10}$$

The value of the polar angle corresponding to the β found above is

$$\vartheta_* = -10°48'12'', \tag{18.11}$$

which value determines the arc of contact L_2. With these values of the unknowns, the left-hand side of relation (18.7) is equal to

$$c(\beta) = 0.99947.$$

By formula (18.10), the maximum pressure is, to a first approximation,**

$$N|_{x=0} \cong \overline{N}_4 = -0.61025P. \tag{18.12}$$

Further, we take $n = 15$ in system (18.4). This system of eight equations (with five separated unknowns in each equation) will be solved by the method of successive approximations, using, as a zero approximation to $\overline{N}_2$, $\overline{N}_4$, $\overline{N}_6$, $\overline{N}_8$, the values (18.10) with the unknown q, respectively.

By solving this system (with β found) and relation (18.7) simultaneously, we obtain (a second approximation according to H. Multhopp)

$$q = -0.52418, \tag{18.13}$$

$$\left.\begin{aligned} \overline{N}_1 &= 0.24397q - 0.14220 = -0.27008, \\ \overline{N}_2 &= 0.48054q - 0.29080 = -0.54269, \\ \overline{N}_3 &= 0.70465q - 0.45269 = -0.82206, \\ \overline{N}_4 &= 0.91061q - 0.63180 = -1.10913, \\ \overline{N}_5 &= 1.09038q - 0.82382 = -1.39537, \\ \overline{N}_6 &= 1.23312q - 1.00927 = -1.65565, \\ \overline{N}_7 &= 1.32624q - 1.15058 = -1.84577, \\ \overline{N}_8 &= 1.35877q - 1.20437 = -1.91663. \end{aligned}\right\} \tag{18.14}$$

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* Here and henceforth, we shall require that the relative error in (18.7) should not exceed 0.1 per cent.

** It will be recalled that the solution $\overline{N(x)}$ corresponding to $n = 7$ is commonly taken to be a first approximation according to H. Multhopp.

For these values of β and $\overline{N}_k$ $(k = 1, \ldots, 8)$ the left-hand side of relation (18.7) gives

$$c(\beta) = 0.99933.$$

For the maximum stress we have

$$N|_{x=0} \cong \overline{N}_8 = -0.61008P. \tag{18.15}$$

As seen from the foregoing, the unknown quantities differ only slightly from each other in a first and a second approximation.

2. Elastic circular disk pressed against a hole in a rigid plate (interior problem). The corresponding equation is obtained from Eq. (15.11) by inserting (17.1) in it:

$$-\frac{\varkappa-1}{\varkappa+1}\frac{\beta}{x^2+\beta^2}N(x)-\frac{1}{2\pi}\int_{-1}^{1}\frac{N'(t)\,dt}{t-x}=\frac{2\beta P}{\pi(1+\varkappa)}\frac{x^2-\beta^2}{(x^2+\beta^2)^2}. \tag{18.16}$$

This equation (subject to the above proviso concerning β) is also identical with Eq. (11.1); this time $B(x) < 0$ on $[-1, 1]$.

System (11.9) for Eq. (18.16) is very little different in form from system (18.4). It has the form

$$\left[-\frac{\varkappa-1}{\varkappa+1}\frac{\beta}{\cos^2\vartheta_m+\beta^2}+b_{mm}\right]\overline{N}_m=\sum_{k=1}^{\frac{1}{2}(n+1)}B_{mk}\overline{N}_k+$$
$$+\frac{2\beta P}{\pi(1+x)}\frac{\cos^2\vartheta_m-\beta^2}{(\cos^2\vartheta_m+\beta^2)^2}\quad\left(m=1,\ldots,\frac{1}{2}(n+1)\right). \tag{18.17}$$

This system must be solved as before by satisfying equality (18.7) and thereby determining the parameter β. When $n = 7$, we have the following values for the unknown quantities ($\overline{N}_k$ are given in terms of P/π):

$$\beta = 0.91612, \tag{18.18}$$

$$\overline{N}_1 = -0.37480,\quad \overline{N}_2 = -0.85751,\quad \overline{N}_3 = -1.47396,$$
$$\overline{N}_4 = -1.84453. \tag{18.19}$$

With these values, $c(\beta) = 1.00034$.

From this the unknown value of the angle ϑ corresponding to the end of the arc of contact is obtained as

$$\vartheta_* = 5°00'46'' \tag{18.20}$$

and the maximum pressure is equal to

$$N|_{x=0} \cong \overline{N}_4 = -0.58713P. \tag{18.21}$$

We further set $n = 15$ in formula (18.17). By solving this system with values (18.18), we obtain

$$\left.\begin{array}{lll} \bar{N}_1 = -0.17990, & \bar{N}_2 = -0.37458, & \bar{N}_5 = -1.15746, \\ \bar{N}_6 = -1.47378, & \bar{N}_3 = -0.59686, & \bar{N}_4 = -0.85727, \\ \bar{N}_7 = -1.73637, & \bar{N}_8 = -1.84215. & \end{array}\right\} \quad (18.22)$$

Then $c(\beta) = 0.99905$. In particular, for the maximum normal stress at a second approximation we have

$$N|_{x=0} \cong \bar{N}_8 = -0.58638P. \qquad (18.23)$$

When $\bar{N}(x)$ and β have been found, it is possible, as indicated above, to calculate all other unknown quantities of the corresponding contact problem.

Note, finally, that with our β defined by formula (18.18) and the adopted value of Poisson's ratio the function $B(x)$ in Eq. (18.16) satisfies condition (12.18) which guarantees the convergence of the method of successive approximations for the solution of system (18.17). However, the method here converges much more slowly than in the case considered in the preceding paragraph. With negative $B(x)$ [even in the presence of condition (12.18)], the method of successive approximations is therefore not always preferable for the solution of system (11.9).

19. Compression of Two Elastic Circular Cylinders of Slightly Different Radii*

The method developed in the preceding sections is also applicable to the case when the radius of a hole in an infinite elastic medium and the radius of a disk (in general, of a different material) inserted into the hole differ from each other. The radius of the hole is taken to be unity and the radius of the disk is denoted by ρ ($\rho < 1$). Suppose that ρ is close to unity.

This time the physical plane is taken to be the plane of the variable $z = x + iy$ and the origin of co-ordinates is placed at the centre of the disk. Let the centre of the hole in the undeformed state be at the point $z_0 = i(1 - \rho)$. Denote by L_1 and L_0 the contours of the hole and disk, respectively.

The compressive forces producing contact between the bodies are assumed to be distributed over the contour of the disk along the upper semi-circumference L_0'. Regarding these forces, we suppose that their resultant of magnitude, say, P passes through the point of initial contact and is directed oppositely to the y axis. As before, the stress and rotation are assumed to vanish at infinity.

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* See the author's paper [42].

The quantities pertaining to the infinite matrix and the disk are identified by the subscripts 1 and 0, respectively.

The field at infinity is absent as before, and hence, for large $|z|$,

$$\Phi_1(z)=\frac{a_1}{z}+O(z^{-2}),\qquad \Psi_1(z)=\frac{b_1}{z}+O(z^{-2}),\tag{19.1}$$

where

$$a_1=\frac{iP}{2\pi(1+\varkappa_1)},\qquad b_1=\frac{i\varkappa_1 P}{2\pi(1+\varkappa_1)}.\tag{19.2}$$

1°. **Boundary conditions of the problem.** Let us connect the normal displacements at points

$$t=\rho e^{i\vartheta}\quad\text{and}\quad t_1=\frac{t}{\rho}+l$$

($l=i(1-\rho)$) lying, respectively, on L_0 and L_1 and making contact with the points of the other body after deformation. Let $r=r_0(\vartheta)$ and $r=r_1(\vartheta)$ be the equations of the curves L_0 and L_1 before deformation, and $r=r(\vartheta)$ the equation of the line of contact (after deformation). The point t with polar co-ordinates (r_0, ϑ) will have moved to a new position $(r_0+v_r^{(0)}, \vartheta+\delta\vartheta)$ after deformation. Similarly, the point t_1 with co-ordinates (r_1, ϑ_1) will occupy a position $(r_1+v_r^{(1)}, \vartheta_1+\delta\vartheta_1)$. By definition, these points must lie on the line of contact and therefore

$$r(\vartheta+\delta\vartheta)=r_0(\vartheta)+v_r^{(0)},\qquad r(\vartheta_1+\delta\vartheta_1)=r_1(\vartheta_1)+v_r^{(1)}.\tag{19.3}$$

If the quantity $1-\rho$ is small, it may be taken that

$$r(\vartheta+\delta\vartheta)=r(\vartheta_1+\delta\vartheta_1)$$

and this, on the basis of formula (19.3), gives an approximate equality

$$v_r^{(0)}-v_r^{(1)}=r_1(\vartheta_1)-r_0(\vartheta).\tag{19.4}$$

On the other hand, as is easily seen,

$$r_0(\vartheta)=\rho,\qquad r_1(\vartheta_1)=\sqrt{1+(1-\rho)^2+2(1-\rho)\sin\vartheta}.$$

Rejecting second-order terms in $1-\rho$ and substituting the value found for r_1 in formula (19.4), we obtain the required equality

$$v_r^{(0)}(\vartheta)-v_r^{(1)}(\vartheta_1)=(1-\rho)(1+\sin\vartheta)\tag{19.5}$$

which holds over the segment of contact.

Denote by γ and γ_1 the parts of the contours L_0 and L_1 along which the elastic bodies will be in contact with each other after deformation. Since the stresses must be continuous in passing from one region to the other, we have

$$N_1(t_1)+iT_1(t_1)=N_0(t)+iT_0(t),\qquad t\in\gamma,\qquad t_1\in\gamma_1,\tag{19.6}$$

where N and T denote, as before, the normal and tangential components of the external stress.

Moreover, we have

$$N_0(t) + iT_0(t) = f_0(t) \quad \text{on} \quad L_0', \tag{19.7}$$

where $f_0 = f_1 + if_2$ is a given function on L_0' which will be assumed sufficiently smooth.

On the remainder of the boundaries L_0' and L_1 free from external effects, we have

$$N = T = 0. \tag{19.8}$$

Finally, by hypothesis, friction is negligible, and hence

$$T_1 = 0 \quad \text{on} \quad \gamma_1, \qquad T_0 = 0 \quad \text{on} \quad \gamma. \tag{19.9}$$

Equalities (19.5) to (19.9) constitute the boundary conditions of our problem.

2°. **Reduction to an integro-differential equation.** The complex potentials $\Phi_1(z)$, $\Psi_1(z)$ satisfy on the boundary of the region the condition (see N. I. Muskhelishvili [93], p. 147)

$$\Phi_1(t) + \overline{\Phi_1(t)} + \left(\frac{dt}{ds}\right)^2 [\bar{t}\Phi_1'(t) + \Psi_1(t)] = N_1 - iT_1, \tag{19.10}$$

where s is the arc length of the circumference L_1 measured from some point.

Introducing new functions $\Phi(\zeta)$ and $\Psi(\zeta)$

$$\Phi(\zeta) = \Phi_1(z), \qquad \Psi(\zeta) = \Psi_1(z), \qquad z = \zeta + l \tag{19.11}$$

which are holomorphic outside $|\zeta| = 1$ and admit at infinity, according to conditions (14.1), the representation

$$\Phi(\zeta) = \frac{a}{\zeta} + O(\zeta^{-2}), \qquad \Psi(\zeta) = \frac{b}{\zeta} + O(\zeta^{-2}), \tag{19.12}$$

we reduce condition (19.10) to

$$\Phi(\sigma) + \overline{\Phi(\sigma)} - \sigma(1 - l\sigma)\Phi'(\sigma) - \sigma^2\Psi(\sigma) = N - iT \quad \text{on} \quad \Gamma. \tag{19.13}$$

In this equality σ denotes a point on the unit circle Γ in the ζ plane and

$$N(\sigma) = N_1(t), \qquad T(\sigma) = T_1(t).$$

From equality (14.13), by virtue of the boundary conditions $T_1 = 0$ on L_1, $N_1 = 0$ on $L_1 - \gamma_1$, and if Eq. (14.12) is also taken into account, similarly to the above, we obtain [see formulas (14.6) and (14.8)]

$$\Phi(\zeta) = -\frac{1}{2\pi i}\int_{\Gamma_1} \frac{N\,d\sigma}{\sigma - \zeta} - \frac{b}{\zeta}, \tag{19.14}$$

$$\zeta(1 - l\zeta)\Phi'(\zeta) + \zeta^2\Psi(\zeta) = b\left(\zeta - \frac{1}{\zeta}\right) + A_1, \quad |\zeta| < 1, \tag{19.15}$$

$$A_1 = -\frac{1}{4\pi i}\int_{\Gamma_1} \frac{N\,d\sigma}{\sigma}, \tag{19.16}$$

where Γ_1 denotes a part of Γ corresponding to the arc γ_1 under translation $z = \zeta + l$. These formulas express the unknown functions Φ_1, Ψ_1 in terms of the unknown values of the normal pressure $N_1(t)$ on the arc γ_1 which is also unknown. For example, from formula (19.14) we have

$$\Phi_1(z) = -\frac{1}{2\pi i}\int_{\gamma_1}\frac{N_1(t)\,dt}{t-z} - \frac{b}{z-l}\,.$$

The boundary relation (19.10) for the elastic disk is of the form

$$\Phi_0(t) + \overline{\Phi_0(t)} - t\Phi_0'(t) - e^{2i\vartheta}\Psi_0(t) = N_0 - iT_0 \text{ on } L_0, \qquad (19.17)$$

where the functions Φ_0, Ψ_0 characterizing the state of stress in the disk are holomorphic in the circle $|z| < \rho$. From relation (19.17), transforming to conjugate values, we obtain

$$\Phi_0(t) + \overline{\Phi_0(t)} - \bar{t}\,\overline{\Phi_0'(t)} - \frac{\bar{t}^2}{\rho^2}\overline{\Psi_0(t)} = N_0 + iT_0 \text{ on } L_0. \quad (19.18)$$

From this, by treatment similar to that from which (19.14) was obtained, we find

$$\Phi_0(z) + \overline{\Phi_0(0)} = \frac{1}{2\pi i}\int_{L_0}\frac{N_0 + iT_0}{t-z}\,dt \quad \text{for } |z| < \rho. \quad (19.19)$$

Without loss in generality we may put $\operatorname{Im}\Phi_0(0) = 0$, and in consequence the right-hand side of formula (19.19) must be real when $z = 0$. The last requirement leads to the condition

$$\int_{L_0'}\frac{f_2\,dt}{t} = 0 \qquad (19.20)$$

which expresses the fact that the resultant couple of the external forces applied to the contour L_0 is zero. This condition is assumed to be fulfilled, and so

$$\Phi_0(z) = \frac{1}{2\pi i}\int_{\gamma}\frac{N_0\,dt}{t-z} - A_0 + F_0(z), \qquad (19.21)$$

$$F_0(z) = \frac{1}{2\pi i}\int_{L_0'}\frac{f_0\,dt}{t-z}, \qquad A_0 = \frac{1}{4\pi i}\int_{L_0}\frac{N_0\,dt}{t}\,. \qquad (19.22)$$

When $\Phi_0(z)$ has been determined, the function $\Psi_0(z)$ is immediately found from the condition

$$t\Phi_0'(t) + \frac{t^2\Psi_0(t)}{\rho^2} - \bar{t}\,\overline{\Phi_0'(t)} - \frac{\bar{t}^2\,\overline{\Psi_0(t)}}{\rho^2} = 2iT_0 \quad \text{on} \quad L_0,$$

which follows from relation (19.18), by the formula

$$z\Phi_0'(z) + \frac{z^2}{\rho^2}\Psi_0(z) = \frac{1}{\pi}\int_{L_0'}\frac{f_2\,dt}{t-z}, \qquad |z| < \rho. \qquad (19.23)$$

The requirement for regularity of the function $\Psi_0(z)$ defined by formula (19.23) leads to the well-known conditions of the statics of elastic bodies. One of these is condition (19.20). As to the second condition

$$\int_{L_0} \frac{N_0 - iT_0}{t^2}\, dt = 0$$

which can easily be reduced to the form

$$\int_{\gamma} N_0(t)\, dt = -P, \tag{19.24}$$

it will serve as an additional condition for determining the unknown quantities N_0 and γ.

We now turn to condition (19.5). By differentiating this equality twice with respect to ϑ and adding to the original one, we have

$$v_r^{(0)}(\vartheta) + \frac{d^2 v_r^{(0)}(\vartheta)}{d\vartheta^2} = v_r^{(1)}(\vartheta_1) + \frac{d^2 v_r^{(1)}(\vartheta_1)}{d\vartheta^2} + 1 - \rho, \quad t \in \gamma, \quad t_1 \in \gamma_1. \tag{19.25}$$

We also write complex representation formulas for displacement components which follow from formulas (1.17) (Chap. I):

$$2\mu_0 v_r^{(0)} = \operatorname{Re}\{e^{i\vartheta}[\varkappa_0 \overline{\varphi_0(t)} - \bar{t}\varphi_0'(t) - \psi_0(t)]\} \quad \text{on} \quad L_0, \tag{19.26}$$

$$2\mu_1 v_r^{(1)} = \operatorname{Re}\{e^{i\alpha}[\varkappa_1 \overline{\varphi_1(t)} - \bar{t}\varphi_1'(t) - \psi_1(t)]\} \quad \text{on} \quad L_1, \tag{19.27}$$

where α is the angle that the inward normal to L_1 at the point t makes with the x axis ($e^{i\alpha} = t - l$) and

$$\varphi_0'(z) = \Phi_0(z),\ \psi_0'(z) = \Psi_0(z),\ \varphi_1'(z) = \Phi_1(z),\ \psi_1'(z) = \Psi_1(z).$$

Setting up the expression on the left-hand side of equality (19.25) for $v_r^{(0)}$ defined by formula (19.26), taking into account relation (19.17) and also the condition $T_0 = 0$ on γ, we obtain

$$2\mu_0\left[v_r^{(0)} + \frac{d^2 v_r^{(0)}}{d\vartheta^2}\right] = \rho \operatorname{Re}\{(\varkappa_0 - 1)\Phi_0(t) - (\varkappa_0 + 1)t\Phi_0'(t) + \\ + 2\Phi_0(t) - N_0(t)\}. \tag{19.28}$$

Similarly, on the basis of formulas (19.27), (19.10) and the condition $T_1 = 0$ on L_1, assuming $t = e^{i\vartheta} + l$ in formula (19.27), we have

$$2\mu_1\left[v_r^{(1)} + \frac{d^2 v_r^{(1)}}{d\vartheta^2}\right] = \operatorname{Re}\{(\varkappa_1 - 1)\Phi_1(t) - (\varkappa_1 + 1)(t - l)\Phi_1'(t) + \\ + 2\Phi_1(t) - N_1(t)\}. \tag{19.29}$$

By virtue of the above two equalities relation (19.25) takes the form

$$\begin{aligned}\rho\mu_1 \operatorname{Re}\{(\varkappa_0-1)\Phi_0(t)-(\varkappa_0+1)t\Phi_0'(t)+2\Phi_0(t)-N_0(t)\} &=\\ =\mu_0\operatorname{Re}\{(\varkappa_1-1)\Phi(\sigma)-(\varkappa_1+1)\sigma\Phi'(\sigma)&+2\Phi(\sigma)-\\ -N(\sigma)\}+2\mu_0\mu_1(1-\rho)\ \text{on}\ \gamma.&\qquad(19.30)\end{aligned}$$

By using, as before, the limiting process formulas in Cauchy-type integrals and taking into account that γ and Γ_1 represent arcs of concentric circles with centre at the origin, from formulas (19.14) and (19.21) we obtain*

$$\left.\begin{aligned}&\operatorname{Re}\{\Phi_0(t_0)-F_0(t_0)\}=\frac{N_0(t_0)}{2}-\frac{1}{4\pi i}\int\limits_{L_0'}\frac{f_1(t)\,dt}{t},\\ &\operatorname{Re}\{t_0[\Phi_0'(t_0)-F_0'(t_0)]\}=\frac{t_0}{2\pi i}\int\limits_{\gamma}\frac{N_0'(t)\,dt}{t-t_0}\quad\text{on}\ \gamma,\end{aligned}\right\}\qquad(19.31)$$

$$\left.\begin{aligned}&\operatorname{Re}\Phi(\sigma_0)=\frac{N(\sigma_0)}{2}-\frac{1}{4\pi i}\int\limits_{\Gamma_1}\frac{N\,d\sigma}{\sigma}+\frac{b}{2}(\sigma_0-\bar{\sigma}_0),\\ &\operatorname{Re}\{\sigma_0\Phi'(\sigma_0)\}=-\frac{\sigma_0}{2\pi i}\int\limits_{\Gamma_1}\frac{N'(\sigma)\,d\sigma}{\sigma-\sigma_0}-\frac{b}{2}(\sigma_0-\bar{\sigma}_0)\quad\text{on}\ \Gamma.\end{aligned}\right\}\qquad(19.32)$$

Substituting the above expressions in relation (19.30) and noting that, by condition (19.6), $N_0(t)=N(\sigma)$ on γ, $t=\rho\sigma$, we obtain after some rearrangement

$$\begin{aligned}kN_0(t_0)+\frac{t_0}{\pi i}\int\limits_{\gamma}\frac{N_0'(t)\,dt}{t-t_0}-\frac{h}{2\pi i}\int\limits_{\gamma}\frac{N_0(t)\,dt}{t}&=pH_0(t_0)+\\ +\frac{qi}{2\pi}P\left(\frac{t_0}{\rho}-\frac{\rho}{t_0}\right)-m(1-\rho)&\quad\text{on}\ \gamma,\qquad(19.33)\end{aligned}$$

where the following notation has been introduced:

$$\left.\begin{aligned}&k=\tau[\mu_0(\varkappa_1-1)-\rho\mu_1(\varkappa_0-1)],\\ &h=\tau\mu_0(\varkappa_1+1),\qquad p=2\tau\rho\mu_1(\varkappa_0+1),\\ &q=2\tau\mu_0\varkappa_1,\qquad m=4\tau\mu_0\mu_1,\\ &\tau=\frac{1}{\mu_0(\varkappa_1+1)+\rho\mu_1(\varkappa_0+1)},\end{aligned}\right\}\qquad(19.34)$$

$$H_0(t)=\operatorname{Re}\left\{F_0(t)-tF_0'(t)-\frac{1}{2}F_0(0)\right\}.\qquad(19.35)$$

This is the required equation for determining N_0 and γ to which must be added condition (19.24). After N_0 and γ are determined, all the unknowns of our problem are found through them in the Cauchy-type integrals.

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* It is assumed *a priori* that arbitrarily small neighbourhoods of the points $z=\rho$, $z=-\rho$ will not be in contact with parts of the external body.

By mapping the circumference $|z| = \rho$ onto the real ξ axis by the function

$$t_0 = i\rho \frac{\xi - i\beta}{\xi + i\beta} \qquad \left(\beta = \frac{\cos\vartheta_*}{1+\sin\vartheta_*}\right) \tag{19.36}$$

so that the arc of contact γ is transformed into the interval $[-1, 1]$, we reduce Eq. (19.33) to the form

$$\frac{k\beta}{\xi^2+\beta^2} N(\xi) - \frac{1}{2\pi}\int_{-1}^{1} \frac{N'(\eta)\,d\eta}{\eta-\xi} - \frac{h\beta^2}{\pi(\xi^2+\beta^2)}\int_{-1}^{1}\frac{N(\eta)\,d\eta}{\eta^2+\beta^2} =$$
$$= \frac{\beta}{\xi^2+\beta^2}\left[pH(\xi) + \frac{qP}{\pi}\frac{\xi^2-\beta^2}{\xi^2+\beta^2} - m(1-\rho)\right] \tag{19.37}$$

with

$$N(1) = N(-1) = 0, \qquad N(\xi) = N_0(t), \qquad H(\xi) = H_0(t).$$

Equality (19.24) then becomes

$$\int_{-1}^{1} \frac{\eta^2-\beta^2}{(\eta^2+\beta^2)^2} N(\eta)\,d\eta = \frac{P}{2\beta\rho}. \tag{19.38}$$

Equations (19.37) and (19.38) are of exactly the same form as the equations considered in the preceding sections. This, in effect, completes the solution of the problem.*

20. On the Generalized Hertz Plane Problem**

In the conditions of the preceding section we reject the assumption that the radii of the hole and disk, which are now denoted by R and r $(r < R)$, are close to each other. It is supposed for ease of computation that the inserted disk is acted on by just one force P applied at its centre and directed as the resultant of the external forces in the preceding section.

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* The problem appears to have been first considered by I. Ya. Shtaerman (see, for example, I. Ya. Shtaerman [128]) who constructs, for its solution, a singular equation similar to that indicated above from entirely different considerations. Having established the approximate equality (19.5), the author proceeds from the formula for the representation of the normal displacement of points of the lip of a (circular) hole in terms of the normal forces distributed along the contour and reduces the problem to equations of the form of (19.37), (19.38) for a function $g(\vartheta)$ related to the normal pressure $N_0(\vartheta)$ by the expression $N_0(\vartheta) = \sec^2\vartheta g'(\tan\vartheta)$. Thereafter these equations are solved by the application of the method of integrating the equation of wing theory presented in the paper of I. N. Vekua [138].

** This section reproduces the author's paper [45] with some modifications.

With such a formulation of the problem, it is natural to expect a small length of contact compared with the dimensions of the bodies and, under an additional assumption that the line of contact is straight, a solution of the problem in closed form (N. I. Muskhelishvili [93], A. V. Bitsadze [7]).

Below, without any assumptions of this kind, we shall attempt to solve the problem approximately by the method developed in the preceding section.

We remark first that the functions $\Phi_1(z)$, $\Psi_1(z)$ have, for large $|z|$, the form defined by formulas (19.1) and the functions $\Phi_0(z)$, $\Psi_0(z)$ admit in the circle $|z| < r$ the representation

$$\Phi_0(z) = \frac{a_0}{z} + \Phi_0^*(z), \qquad \Psi_0(z) = \frac{b_0}{z} + \Psi_0^*(z), \tag{20.1}$$

where

$$a_0 = \frac{iP}{2\pi(1+\varkappa_0)}, \qquad b_0 = \varkappa_0 a_0 \tag{20.2}$$

and Φ_0, Ψ_0 are holomorphic in the same circle.

Let

$$y = f_1(x), \qquad y = f_0(x)$$

be the equations of the curves L_1 and L_0. If v_1 and v_0 are the projections on the y axis of the displacements at points (x, f_1) and (x, f_0) lying, respectively, on L_1 and L_0 before deformation, then after deformation we have on the segment of contact (see N. I. Muskhelishvili [93], p. 437)

$$v_0 - v_1 = f_1(x) - f_0(x).$$

From this, taking into account that on a segment of the x axis corresponding to the line of contact γ

$$f_0(x) = -\sqrt{r^2 - x^2}, \qquad f_1(x) = R - r - \sqrt{R^2 - x^2},$$

we obtain, neglecting small quantities of higher order, the equality

$$v_0 - v_1 = \frac{1}{2}\frac{r-R}{rR}x^2 \tag{20.3}$$

which may also be written as*

$$v_0 - v_1 = \frac{1}{8}\frac{r-R}{rR}(t+\bar{t})^2 \qquad (t \text{ on } \gamma, \quad t = re^{i\vartheta}). \tag{20.3a}$$

This condition replaces, in our case, the boundary condition (19.5) of the preceding section. The other boundary conditions remain unaltered if the right-hand side of formula (19.7) is replaced by zero. In particular, conditions (19.8) are fulfilled everywhere in parts L_0' and L_1' not coming into contact after deformation.

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* This equality is only approximate in the general case of $r < R$. It is true to a high degree of accuracy when the ratio r/R is small compared with unity.

Disregarding, for the present, condition (20.3), we shall determine, as before, the complex potentials in terms of undetermined normal pressures beneath the disk.

We have

$$\Phi_1(z) = -\frac{1}{2\pi i}\int\limits_{\gamma}\frac{N_1(t)\,dt}{t-z} + F_1(z), \tag{20.4}$$

$$\Phi_0(z) = \frac{1}{2\pi i}\int\limits_{\gamma}\frac{N_0(t)\,dt}{t-z} + F_0(z) + A, \tag{20.5}$$

$$F_1(z) = -\frac{b_1}{z-l}, \quad F_0(z) = \frac{a_0}{z} + \frac{2a_0 z}{r^2}, \tag{20.6}$$

where

$$A = -\frac{1}{4\pi i}\int\limits_{\gamma}\frac{N_0(t)\,dt}{t}.$$

Turning to condition (20.3), we write the well-known formulas for the general complex representation of elastic displacements [formula (1.3), Chap. I]

$$2\mu_0(u_0 + iv_0) = \varkappa_0\varphi_0(t) - t\overline{\varphi_0'(t)} - \overline{\psi_0(t)} \quad \text{on} \quad L_0, \tag{20.7}$$

$$2\mu_1(u_1 + iv_1) = \varkappa_1\varphi_1(t) - t\overline{\varphi_1'(t)} - \overline{\psi_1(t)} \quad \text{on} \quad L_1 \tag{20.8}$$

and recall that the complex potentials Φ, Ψ characterizing the stresses in the disk and the infinite plate satisfy conditions (19.17) and (19.10) on the corresponding boundaries.

By differentiating the above equalities with respect to the arc length s along the contact line γ and taking into account formulas (19.10) and (19.17), we find

$$\left.\begin{aligned} 2r\mu_0\frac{dv_0}{ds} &= \operatorname{Re}\{t\,[(\varkappa_0+1)\,\varphi_0'(t) - N_0(t)]\},\\ 2r\mu_1\frac{dv_1}{ds} &= \operatorname{Re}\{t\,[(\varkappa_1+1)\,\varphi_1'(t) - N_1(t)]\} \quad \text{on} \quad \gamma. \end{aligned}\right\} \tag{20.9}$$

The consideration of the boundary condition (20.3) on the basis of the above equalities now gives the relation

$$\operatorname{Re}\{\mu_1 t\,[(\varkappa_0+1)\,\Phi_0(t) - N_0(t)] - \mu_0 t\,[(\varkappa_1+1)\,\Phi_1(t) - N_1(t)]\} = h(t),$$

where

$$h(t) = \frac{i}{2}\mu_0\mu_1\frac{r-R}{rR}(t^2 - \bar{t}^2) \quad \text{on} \quad \gamma. \tag{20.10}$$

For the limiting values $\Phi_0(t)$, $\Psi_0(t)$ we use, as before, the Sokhotskii-Plemelj formula and note that on the line γ representing an arc of the circle L_0 the following equalities hold: $N_0(t) = N_1(t)$,

$$\operatorname{Re}\frac{t_0}{2\pi i}\int\limits_{\gamma}\frac{N_0(t)\,dt}{t-t_0} = \frac{t_0-\bar{t}_0}{4\pi i}\int\limits_{\gamma}\frac{N_0(t)\,dt}{t-t_0} + \frac{\bar{t}_0}{4\pi i}\int\limits_{\gamma}\frac{N_0(t)\,dt}{t}.$$

From relation (20.10) we then obtain after some rearrangement

$$\frac{k_0}{4}(t_0+\bar{t}_0)\,N_0(t_0)+\frac{p_0(t_0-\bar{t}_0)}{4\pi i}\int_\gamma\frac{N_0(t)\,dt}{t-t_0}+$$
$$+\frac{(2p_0-q_0)\,\bar{t}_0-q_0t_0}{8\pi i}\int_\gamma\frac{N_0(t)\,dt}{t}+\operatorname{Re}\{t_0\,[q_0F_0(t_0)-$$
$$-m_0F_1(t_0)]\}=h(t_0)\quad\text{on}\quad\gamma,\tag{20.11}$$

where

$$\left.\begin{aligned}k_0&=\mu_1(\varkappa_0-1)-\mu_0(\varkappa_1-1), & p_0&=\mu_1(\varkappa_0+1)+\mu_0(\varkappa_1+1),\\ q_0&=\mu_1(\varkappa_0+1), & m_0&=\mu_0(\varkappa_1+1),\end{aligned}\right\}\tag{20.12}$$

and F_0, F_1 are known functions given by formulas (20.6).

This is the required integral equation for the unknown pressure $N_0(t)$. To this equation must be added, as before, relation (19.24) for the determination of the arc of contact

$$\int_\gamma N_0(t)\,dt=-P,\tag{20.13}$$

a relation which expresses the condition for the equilibrium of the elastic disk pressed against the hole by a concentrated force. Moreover, here, too, the following conditions are fulfilled:

$$N_0(t_*)=N_0(-\bar{t}_*)=0,\qquad t_*=re^{i\vartheta_*},\tag{20.14}$$

where t_* and $-\bar{t}_*$ are the affixes of the end points of the line γ which lie to the right and left of the y axis, respectively.

Substitution (19.36) with $\rho=r$ reduces these equations to the form

$$\frac{k\beta\xi}{\xi^2-\beta^2}N(\xi)+\frac{1}{2\pi}\int_{-1}^{1}\frac{N(\eta)\,d\eta}{\eta-\xi}+\frac{m\beta^2\xi}{\pi(\xi^2-\beta^2)}\int_{-1}^{1}\frac{N(\eta)\,d\eta}{\eta^2+\beta^2}=\frac{\xi^2+\beta^2}{\xi^2-\beta^2}F(\xi),\tag{20.15}$$

$$\frac{2\beta r}{P}\int_{-1}^{1}\frac{\xi^2-\beta^2}{(\xi^2+\beta^2)^2}N(\xi)\,d\xi=1,\tag{20.16}$$

where

$$\left.\begin{aligned}&F(\xi)=\frac{1}{rp_0}\{h(t_0)-\operatorname{Re}[t_0(q_0F_0(t_0)-m_0F_1(t_0))]\},\\ &k=\frac{k_0}{p_0},\quad m=\frac{m_0}{p_0},\quad N(\xi)=N_0(t_0),\quad N(1)=N(-1)=0.\end{aligned}\right\}\tag{20.17}$$

Thus, to determine the unknowns N and β we have the singular integral equation (20.15) with the additional condition (20.16).

After N and β are determined, the stress functions $\Phi_0(z)$ and $\Phi_1(z)$ are given by formulas (20.4) and (20.5). As far as the functions $\Psi_0(z)$ and $\Psi_1(z)$ are concerned, they are found directly from the functions Φ_0 and Ψ_0 by the boundary conditions of the problem, as in the preceding section.

The singular equation (20.15) belongs to the type of equations considered in Sec. 13* and, according to the condition $N(1) = N(-1) = 0$, the procedure of numerical solution presented there in Par. 3° applies to it.

Equation (20.15) is written in the form of (13.22). We have

$$a(\xi)N(\xi)+\frac{b(\xi)}{2\pi}\int_{-1}^{1}\frac{N(\eta)\,d\eta}{\eta-\xi}+\frac{1}{2\pi}\int_{-1}^{1}k(\xi,\eta)N(\eta)\,d\eta=f(\xi), \tag{20.18}$$

where

$$\left.\begin{aligned} &a(\xi)=\frac{k\beta\xi}{\xi^2-\beta^2}, \qquad b(\xi)=1, \qquad k(\xi,\eta)=\frac{2m\beta^2\xi}{(\xi^2-\beta^2)(\eta^2+\beta^2)},\\ &f(\xi)=\frac{4\beta\mu_1}{p_0}\left[\mu_0\left(1-\frac{r}{R}\right)+\frac{P}{\pi r}\right]+\\ &\qquad+\frac{m\beta P\xi(R-r)(\xi^2+\beta^2)}{\pi(\varkappa_0+1)(\xi^2-\beta^2)[\xi^2(R-2r)^2+\beta^2R^2]}.\end{aligned}\right\} \tag{20.19}$$

According to the analysis developed in Par. 3° of Sec. 13, the interpolation nodes in our case are the points

$$x_m=\cos\vartheta_m, \qquad \vartheta_m=\frac{m\pi}{n+1} \qquad (m=1,\ldots,n)$$

and the system of linear equations for discrete values of the unknown function [system (13.10)] is of the form

$$a_m\bar{N}_m=f_m+\sum_{\nu=1}^{n}\alpha_{m,\nu}\bar{N}_\nu \qquad (m=1,\ldots,n), \tag{20.20}$$

where

$$\alpha_{m,\nu}=\frac{\sin\vartheta_\nu}{n+1}\left[\frac{\varepsilon_{m,\nu}}{\cos\vartheta_m-\cos\vartheta_\nu}-\frac{1}{2}k(\cos\vartheta_m,\cos\vartheta_\nu)\right],$$

$$\varepsilon_{m,\nu}=\begin{cases}1 & \text{when } |m-\nu|=1,3,\ldots,\\ 0 & \text{when } |m-\nu|=0,2,\ldots.\end{cases}$$

We finally observe that, by the conditions of the problem, the function $N(\xi)$ is even, and in consequence the number of equations in system (20.20) is halved. It contains altogether $\left[\frac{n+1}{2}\right]$ unknowns.

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* See the note at the end of the section.

Numerical example. By way of example, we suppose that the circular disk is made of steel, and the surrounding body of cast iron. Accordingly we accept the following values for the elastic constants:

$$\nu_0 = 0.33,\ \nu_1 = 0.27, \qquad \varkappa_0 = 3 - 4\nu_0 = 1.67, \qquad \varkappa_1 = 3 - 4\nu_1 = 1.92.$$

$$\mu_0 = \frac{E_0}{2(1+\nu_0)} = 8.1 \times 10^5 \text{ kgf/cm}^2, \qquad \mu_1 = \frac{E_1}{2(1+\nu_1)} = 4.5 \times 10^5 \text{ kgf/cm}^2.$$

We further take $P = 1$ kgf and $r = 3$ cm. The calculations will be carried out when the ratio r/R is, respectively, 0.9, 0.5 and 0.

In system (20.20) we take $n = 8$, i.e., we solve a system of four equations in N_1, N_2, N_3, N_4; $\vartheta_\nu = \frac{\nu\pi}{9}$ $(\nu = 1, 2, 3, 4)$.

The results of the calculations are presented in the following table:

Table 1

r/R	0.9	0.5	0
$\overline{N}_1$	—1.00402	—2.24803	—3.18046
$\overline{N}_2$	—1.89353	—4.23317	—5.98690
$\overline{N}_3$	—2.55914	—5.71031	—8.07370
$\overline{N}_4$	—2.91538	—6.49854	—9.18564
ϑ_*	—85°52′20″	—88°09′20″	—88°41′40″

For the resulting $\overline{N}_\nu$ and β*, the left-hand side of expression (20.16) designated by $c(\beta)$, being calculated with the use of the Gauss-type quadrature formula

$$c(\beta) = \frac{2r\pi\beta}{n+1} \sum_{\nu=1}^{n} \frac{\sin\vartheta_\nu(\cos^2\vartheta_\nu - \beta^2)}{(\cos^2\vartheta_\nu + \beta^2)^2} \frac{\overline{N}_\nu}{P},$$

has the following values:

$c(\beta) = 1.00143$ when $r/R = 0.9$,

$c(\beta) = 0.99996$ when $r/R = 0.5$,

$c(\beta) = 0.99997$ when $r/R = 0$.

It appears from these data that equality (20.16) is fulfilled with high accuracy in all three cases.

As noted at the beginning of the section, the problem considered can be solved in closed form under certain assumptions. The pressure $N(x)$ and the region of contact γ which in this case represents the

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* It will be recalled that β is determined in terms of ϑ_* by formula (19.36).

segment $[-l, l]$ of the x axis are given by the formulas (N. I. Muskhelishvili [93], p. 438)*

$$-N(x)=\frac{\sqrt{l^2-x^2}}{\pi K}\int_{-l}^{l}\frac{g'(t)}{\sqrt{l^2-t^2}}\frac{dt}{t-x},\qquad \int_{-l}^{l}\frac{tg'(t)\,dt}{\sqrt{l^2-t^2}}=KP$$

with

$$g(x)=v_1(x)-v_0(x),\qquad K=\frac{\varkappa_0+1}{4\mu_0}+\frac{\varkappa_1+1}{4\mu_1}.$$

But in our case, according to formula (20.3a),

$$g(x)=\frac{1}{2}\left(\frac{1}{r}-\frac{1}{R}\right)x^2,$$

and in consequence the above integrals are evaluated in an elementary way. We have

$$N(x)=\frac{r-R}{rRK}\sqrt{l^2-x^2},\qquad l^2=\frac{2\pi rR}{R-r}KP.$$

For comparison we present the numerical values of the stresses and the polar angles obtained from the above formulas with the same parameters of the problem.

Table 2

r/R	0.9	0.5	0
$\overline{N}_1$	—1.00729	—2.25223	—3.18532
$\overline{N}_2$	—1.89324	—4.23315	—5.98690
$\overline{N}_3$	—2.55063	—5.70303	—8.06575
$\overline{N}_4$	—2.90053	—6.48538	—9.17223
ϑ_*	—85°52′19″	—88°09′14″	—88°41′14″

The difference between the data of Tables 1 and 2 is negligibly small.

21. On a Generalization of the Method

The method of solving contact problems developed in the present part is generalized to the case of non-circular holes of a certain class. This applies, in the first place, to holes which are reducible to a circular one when the region is mapped by a function of the form

$$\omega(\zeta)=R\left(\zeta+\frac{m}{\zeta^n}\right), \tag{21.1}$$

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* The formulas are presented in the symmetrical case.

where R, m, n are some constants. By a suitable choice of these constants it is possible, with the use of formula (21.1), to consider holes in the form of a circle, ellipse, curvilinear quadrilateral, etc.

In solving the problem of a rigid inclusion in these cases it is necessary, after first using mapping (21.1), to employ the method developed in Sec. 14, which leads to a singular equation of the form of (14.12), though somewhat more complicated.* This equation may be successfully solved by Multhopp's method or by any other approximate method. In this way the problem of a rigid elliptical inclusion was solved in the work of H. B. Wilson, Jr. [149] and J. G. Goree [30].

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* The reduction of the problem to an integro-differential equation can be effected in a somewhat different way by the application of the method of I. N. Kartsivadze (see the note at the end of Sec. 14).

CHAPTER FOUR

Special Methods for the Solution of Plane Problems

The chapter is concerned with several topics of the mathematical theory of elasticity and contains solutions of various specific problems. In the first section an efficient method of solving plane problems is proposed for a rather wide class of singly connected regions with sectionally smooth boundaries. The second section gives a closed solution of the problem of the reinforcing of thin plates with straight stiffeners of uniform section. In the third section a new procedure is indicated for determining the stress intensity near holes and, finally, in the fourth section one of the well-known methods of solving the plane problem is realized in numerical form.

I. SOLUTION OF FUNDAMENTAL PROBLEMS FOR ONE CLASS OF SINGLY CONNECTED REGIONS

A class of regions which will be considered in the present part is defined by the condition that the function $\omega(\zeta)$ giving a conformal mapping of a region S onto a semi-circle (onto a half-plane with a cut-out in the form of a semi-circle) should admit an analytic continuation to the whole of the unit circle (to the infinite region outside the circular hole). After a suitable continuation of either of the unknown complex potentials, the plane problem for the region S is reduced to some mixed problem for determining a single function holomorphic inside the unit circle (in the plane with the circular hole). For such regions, we shall indicate a method of solving fundamental plane problems which furnishes the means of reducing them to an infinite system of linear algebraic equations.

22. Method of Solution

Let Σ^+ and Σ^- denote the upper and lower semi-circles of unit radius in the ζ plane (with centre at the origin), γ^+ and γ^- the curved parts of their boundaries, respectively, and l the interval $(-1, 1)$ in the same plane.

After mapping the physical region S onto the lower semi-circle Σ^-, the boundary condition of the plane problem is represented in the form [Chap. I, formula (1.14)]

$$\varphi(\sigma)+\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\varphi'(\sigma)}+\overline{\psi(\sigma)}=f(\sigma) \quad \text{on} \quad \gamma^{-}+l, \tag{22.1}$$

where $f(\sigma)$ is a given function. We require of this function that its values on the interval l be real and represent boundary values of some function $f_0(z)$ holomorphic in Σ^{-}. It is assumed that condition (22.1) is fulfilled everywhere on $\gamma^{-}+l$, with the possible exception of the angular points.

We extend the definition of the complex potential $\varphi(\zeta)$ to the region Σ^{+} by the equality

$$\varphi(\zeta)=-\omega(\zeta)\frac{\bar{\varphi}'(\zeta)}{\bar{\omega}'(\zeta)}-\bar{\psi}(\zeta)+\bar{f}_0(\zeta) \qquad (\zeta \text{ in } \Sigma^{+}), \tag{22.2}$$

where use has been made of the notation (N. I. Muskhelishvili [93], Sec. 76)

$$\bar{F}(\zeta)=\overline{F(\bar{\zeta})}. \tag{22.3}$$

It can easily be verified that the function $\varphi(\zeta)$ defined by equality (22.2) is holomorphic in the region Σ^{+} and, according to (22.1), satisfies the condition

$$\varphi^{+}(\sigma)=\varphi^{-}(\sigma) \qquad \text{on } l. \tag{22.4}$$

We replace the variable ζ by $\bar{\zeta}$ in formula (22.2) (assuming that ζ belongs to Σ^{-}) and transform to conjugate values on both sides of the equality. We have

$$\psi(\zeta)=-\bar{\varphi}(\zeta)-\frac{\bar{\omega}(\zeta)\,\varphi'(\zeta)}{\omega'(\zeta)}+f_0(\zeta) \qquad (\zeta\in\Sigma^{-}). \tag{22.5}$$

The above equality makes it possible to determine the potential $\psi(\zeta)$ in terms of the function $\varphi(\zeta)$ holomorphic, by condition (22.4), in the entire unit circle.

The boundary condition (22.1) on the interval l has already been used. If the function $\overline{\psi(\sigma)}$ in the remainder of this condition is replaced by its value from formula (22.5), we obtain

$$\varphi(\sigma)-\varphi(\bar{\sigma})+[\omega(\sigma)-\omega(\bar{\sigma})]\overline{\Phi(\sigma)}=f_{*}(\sigma) \quad \text{on} \quad \gamma^{-}, \tag{22.6}$$

where

$$f_{*}(\sigma)=f(\sigma)-\overline{f_0(\sigma)}, \qquad \Phi(\sigma)=\frac{\varphi'(\sigma)}{\omega'(\sigma)}.$$

Thus, for the determination of the single function $\varphi(\zeta)$ holomorphic in the circle $|\zeta|<1$ we have condition (22.6) on the lower semi-circumference. The condition on the upper semi-circumference is obtained from equality (22.6) by replacing the point σ by $\bar{\sigma}$

in it. We have

$$\varphi(\sigma)-\varphi(\bar{\sigma})+[\omega(\sigma)-\omega(\bar{\sigma})]\,\overline{\Phi}(\sigma)=-f_*(\bar{\sigma}) \quad \text{on } \gamma^+. \tag{22.6a}$$

By combining formulas (22.6) and (22.6a) into one boundary condition, we arrive at the following mixed problem of the theory of analytic functions.

Find a holomorphic function $\varphi(\zeta)$ *in a unit circle from the boundary condition*

$$\varphi(\sigma)-\varphi(\bar{\sigma})+[\omega(\sigma)-\omega(\bar{\sigma})]\,G(\sigma)=g(\sigma) \quad \text{on } \gamma \quad (\gamma=\gamma^-+\gamma^+), \tag{22.7}$$

$$G(\sigma)=\begin{cases}\overline{\Phi(\sigma)} & \text{on } \gamma^-,\\ \overline{\Phi}(\sigma) & \text{on } \gamma^+,\end{cases} \qquad g(\sigma)=\begin{cases} f_*(\sigma) & \text{on } \gamma^-,\\ -f_*(\bar{\sigma}) & \text{on } \gamma^+,\end{cases} \tag{22.8}$$

where $f_*(\sigma)$ is a given function on γ^- and the $\omega(\sigma)$ are the boundary values of a given function $\omega(\zeta)$ holomorphic inside γ.

Under certain conditions the power series method can be applied to the boundary value problem (22.7); this leads to an infinite system of linear algebraic equations in the unknown coefficients in the expansion of the function $\varphi(\zeta)$ into a power series.

The foregoing method is easily extended to the case of regions which can be mapped onto a circular half-ring by means of power series.

We shall not elaborate upon detailed computation and analysis of the solution in the general case. The application of the method will be illustrated below by specific examples.

23. Fundamental Mixed Problem for a Semi-circle*

Consider the problem of the equilibrium of an elastic semi-circle pressed, under conditions of plane strain (or plane stress), against an absolutely rigid profile with a straight base. It is assumed that contact between the elastic body and the rigid punch takes place along the diameter of the circle and the external forces acting on the body (and, of course, holding it in a state of equilibrium) are distributed along the semi-circumference according to a given law.

We take the radius of the semi-circle to be unity and place the elastic body with the punch in the plane $z=x+iy$ so that the elastic medium occupies the lower half of the circle with centre at the origin of co-ordinates. Clearly, in the case under consideration

$$z=\omega(\zeta)=\zeta$$

The upper and lower semi-circles in the z plane are denoted, respectively, by S^+ and S^-, while the notation for their curved boundaries and also for the interval $-1<z<1$ is left unchanged.

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* The present section reproduces the author's paper [48] with slight modifications.

Suppose that there is no slipping or separation on the line of contact between two bodies (the elastic semi-circle and the rigid base).* For the complex potentials $\varphi(z)$ and $\psi(z)$ holomorphic in the region S^-, we then have the boundary conditions

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = f(t) \qquad \text{on } \gamma^-, \tag{23.1}$$

$$\varkappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)} = 0 \qquad \text{on } l. \tag{23.2}$$

Here $f(t)$ is a given function of the point on the lower semi-circumference, continuous on the closed arc, and $\varkappa$ is a constant equal to $3-4\nu$ in the case of plane strain and to $\frac{3-\nu}{1+\nu}$ in the case of plane stress. The boundary conditions (23.1), (23.2) are fulfilled everywhere on the corresponding unclosed curves, with the possible exception of the end points $z=\pm1$.

Following the procedure indicated in the preceding section, we define the function $\varphi(z)$ in the region S^+ by the relation

$$\varkappa\varphi(z) = z\bar{\varphi}'(z) + \bar{\psi}(z) \qquad (z \text{ in } S^+). \tag{23.3}$$

If we replace z by $\bar{z}$ in formula (23.3) (assuming that z belongs to S^-) and transform to conjugate values on both sides of the equality, then this equality determines $\psi(z)$ in terms of the function $\varphi(z)$ holomorphic in both semi-circles

$$\psi(z) = \varkappa\bar{\varphi}(z) - z\varphi'(z) \qquad (z \text{ in } S^-). \tag{23.4}$$

It can easily be verified that the boundary condition (23.2) is nothing but the condition for the analytic continuation of the function $\varphi(z)$ through the interval l. For the function φ which is holomorphic, from the above, in the entire unit circle we have, on the basis of formulas (23.1) and (23.4), the following condition:

$$\varphi(t) + \varkappa\varphi(\bar{t}) + (t-\bar{t})\,\overline{\varphi'(t)} = f(t) \qquad \text{on } \gamma^-. \tag{23.5}$$

By continuing, as before, equality (23.5) to the upper semi-circumference γ^+, we obtain relations for determining the function $\varphi(z)$ on the entire circumference γ:

$$\varphi(t) + \varkappa\varphi(\bar{t}) + (t-t)\overline{\varphi'(t)} = g(t) + k(t)\,\Omega(t) \tag{23.6}$$

$$g(t) = f(t) \ \text{ on } \gamma^-, \qquad g(t) = f(\bar{t}) \ \text{ on } \gamma^+, \tag{23.7}$$

$$\Omega(t) = (\varkappa-1)\,[\varphi(\bar{t}) - \varphi(t)] + (t-\bar{t})\,[\overline{\varphi'(t)} + \bar{\varphi}'(t)], \qquad k(t) = \begin{cases} 0 & \text{on } \gamma^-, \\ 1 & \text{on } \gamma^+. \end{cases} \tag{23.8}$$

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* The proposed method is suitable, in principle, in other cases provided that the coefficient of friction retains the same constant value along the contact line-

We shall apply the power series method to the solution of the mixed problem (23.6). Assume in the unit circle

$$\varphi(z)=\sum_{k=0}^{\infty} a_k z^k, \qquad \varphi'(z)=\sum_{k=1}^{\infty} k a_k z^{k-1}. \tag{23.9}$$

Supposing the legitimacy of the corresponding expansions into Fourier series, we further assume

$$g(t)=\sum_{k=-\infty}^{\infty} A_k t^k, \qquad \Omega(t)=\sum_{k=-\infty}^{\infty} \Lambda_k t^k \quad (t=e^{i\vartheta}). \tag{23.10}$$

The quantities A_k are known. The Fourier coefficients Λ_k of the unknown function $\Omega(t)$ are represented, according to (23.8), by the formulas

$$\Lambda_n=\frac{1}{2\pi i}\int_0^{\pi} \{(\varkappa-1)[\varphi(\bar t)-\varphi(t)]+(t-\bar t)[\overline{\varphi'(t)}+\overline{\varphi}'(t)]\}\, t^{-n-1}\,dt. \tag{23.11}$$

The character of the desired solution in the closed region must be determined from the requirement that the energy integral be finite. To fulfil this condition, it is sufficient to assume that *the field of elastic displacements is continuous in the closed region and the stress components are also everywhere continuous up to the boundary, with the exception of the points* $z=\pm 1$; *the stresses in the neighbourhood of these points may admit a singularity of order less than unity* (see, for example, N. I. Muskhelishvili [92], Secs. 113 to 115). Under these conditions it may be stated that the derivatives $\varphi'(z)$, $\varphi''(z)$ are continuous in any part of the closed region not containing the points $z=\pm 1$, and near these points the following estimate is valid:

$$|\varphi'(z)| < C\,|z\pm 1|^{-\alpha} \qquad (0 \leqslant \alpha < 1,\ C=\text{constant}). \tag{23.12}$$

Revert to the solution of problem (23.6). Setting up an expression for $\Omega(t)$ on the basis of expansions (23.9), we have

$$\left.\begin{aligned}\Omega(t)&=\sum_{k=1}^{\infty}\Omega_k[a;\varkappa](t^k-t^{-k})+\bar a_1(t-t^{-1}),\\ \Omega_k[a;\varkappa]&=-[(\varkappa-1)a_k+(k+2)\bar a_{k+2}-k\bar a_k].\end{aligned}\right\} \tag{23.13}$$

On the basis of formula (23.11) we further have

$$\Lambda_0 = \sum_{k=1}^{\infty} a_{k0}\Omega_k\,[a;\ \varkappa] - \frac{2\bar{a}_1}{\pi i}, \tag{23.14}$$

$$\Lambda_n = \frac{1}{2}(\operatorname{sgn} n)\,\{\bar{a}_1 + \Omega_{|n|}\,[a;\ \varkappa]\} + \sum_{k=2}^{\infty} a_{k1}\Omega_k\,[a;\ \varkappa], \quad n = \pm 1, \tag{23.15}$$

$$\Lambda_n = \frac{1}{2}(\operatorname{sgn} n)\,\Omega_{|n|}\,[a;\ \varkappa] + a_{1n}\bar{a}_1 + {}$$

$$+ \sum_{k=1}^{\infty}{}' a_{kn}\Omega_k\,[a;\ \varkappa], \qquad n = \pm 2,\ \pm 3,\ \ldots. \tag{23.16}$$

Here

$$a_{kn} = \frac{1}{2\pi i}\left[\frac{(-1)^{k-n}-1}{k-n} + \frac{(-1)^{k+n}-1}{k+n}\right], \qquad \begin{matrix} k = 1,\ 2,\ \ldots, \\ n = 0,\ \pm 1,\ \pm 2,\ \ldots. \end{matrix}$$

A prime on the summation denotes that the value $k = |\,n\,|$ is omitted in the summation.

Set up the left-hand side of equality (23.6) with the use of series (23.9). We obtain

$$\varphi(t) + \varkappa\varphi(\bar{t}) + (t - \bar{t})\,\overline{\varphi'(t)} = (1+\varkappa)\,a_0 + 2\bar{a}_2 + \bar{a}_1 t + {}$$

$$+ \sum_{k=1}^{\infty} a_k t^k - \sum_{k=1}^{\infty} \Omega_k\,[a;\ \varkappa+1]\,t^{-k}. \tag{23.17}$$

We now insert series (23.17), (23.10) in the boundary condition (23.6). By comparing the coefficients of t^n ($n = 0,\ 1,\ 2,\ \ldots$), we obtain successively, using formulas (23.14) to (23.16),

$$\left.\begin{aligned} &(1+\varkappa)\,a_0 + 2\bar{a}_2 = A_0 + \sum_{k=1}^{\infty} a_{k0}\Omega_k\,[a;\ \varkappa] - \frac{2\bar{a}_1}{\pi i}, \\ &a_1 + \frac{1}{2}\bar{a}_1 = A_1 + \frac{1}{2}\Omega_1\,[a;\ \varkappa] + \sum_{k=2}^{\infty} a_{k1}\Omega_k\,[a;\ \varkappa], \\ &a_n = A_n + a_{1n}\bar{a}_1 + \frac{1}{2}\Omega_n\,[a;\ \varkappa] + \sum_{k=1}^{\infty}{}' a_{kn}\Omega_n\,[a;\ \varkappa] \quad (n \geqslant 2). \end{aligned}\right\} \tag{23.18}$$

The Fourier coefficients of the function $g(t)$, as appears from its very definition (23.7), are related by the equalities $A_n = A_{-n}$ ($n = 1,\ 2,\ \ldots$).

For this reason the comparison of the coefficients of the negative powers of t in formula (23.6) does not give any new equations [different from (23.18)].

The infinite system of linear equations (23.18) thus represents a complete system of equations for determining the unknown coefficients in expansions (23.9). After finding the solution a_k ($k = 0, 1, \ldots$) of this system, the function $\varphi(z)$ gives (provided the corresponding series converge uniformly) the solution of the formulated problem.

Having determined $\varphi(z)$ in the circle, we find the function $\psi(z)$ by formula (23.4). After that all the unknown elements of the displacement and stress fields can be found in the usual way. In particular, for the stress σ_y defined by the formula

$$\sigma_y = \operatorname{Re}\{\varphi'(z) + \overline{\varphi'(z)} + z\overline{\varphi''(z)} + \overline{\psi'(z)}\},$$

we have, on the basis of formula (23.4),

$$\sigma_y = \operatorname{Re}\{\varphi'(z) + \varkappa\overline{\varphi'(z)} + (z - \bar{z})\overline{\varphi''(z)}\}.$$

On the interval l this expression takes the form

$$\sigma_y = \operatorname{Re}\{\varphi'(z) + \varkappa\overline{\varphi'(z)}\} = \operatorname{Re}\sum_{k=1}^{\infty}(1+\varkappa)\,ka_k x^{k-1} \qquad (-1 < x < 1). \quad (23.19)$$

Let us introduce into relation (23.18) new unknowns b_k defined by the equalities

$$b_0 = a_0, \qquad b_k = ka_k \qquad (k = 1, 2, \ldots)$$

and represent the system of equations in expanded form. We obtain

$$(1+\varkappa)\,b_0 = -\bar{b}_2 + \frac{2}{\pi i}\sum_{k=1}^{\infty}\frac{1}{2k-1}\left[\frac{\varkappa-1}{2k-1}b_{2k-1} + \bar{b}_{2k+1} - \bar{b}_{2k-1}\right] + A_0, \quad (23.20')$$

$$\frac{1+\varkappa}{2n-1}b_{2n-1} = \bar{b}_{2n-1} - \bar{b}_{2n+1} + \\ + \frac{2}{\pi i}\sum_{k=1}^{\infty}\frac{4k}{4k^2-(2n-1)^2}\left[\frac{\varkappa-1}{2k}b_{2k} + \bar{b}_{2k+2} - \bar{b}_{2k}\right] + 2A_{2n-1}, \quad (23.20'')$$

$$\frac{1+\varkappa}{2n}b_{2n} = \bar{b}_{2n} - \bar{b}_{2n+2} + \frac{4}{\pi i}\,\frac{\bar{b}_1}{4n^2-1} + \\ + \frac{2}{\pi i}\sum_{k=1}^{\infty}\frac{4k-2}{(2k-1)^2-4n^2}\left[\frac{\varkappa-1}{2k-1}b_{2k-1} + \bar{b}_{2k+1} - \bar{b}_{2k-1}\right] + 2A_{2n} \\ (n = 1, 2, \ldots). \quad (23.20''')$$

Here and henceforth, the first term on the right-hand side of the second equality must be left out when $n = 1$.

We shall not elaborate upon the analysis of the system of equations (23.20), but point the way to its approximate solution. The approximate solution of the system of equations (23.20) is conveniently found from the corresponding truncated system of equations

$$(1+\varkappa) b_0 = -\bar{b}_2 + \frac{2}{\pi i}\sum_{k=1}^{N}\frac{1}{2k-1}\left[\frac{\varkappa-1}{2k-1} b_{2k-1} + \bar{b}_{2k+1} - \bar{b}_{2k-1}\right] + A_0, \tag{23.21$'$}$$

$$\frac{1+\varkappa}{2n-1} b_{2n-1} = \bar{b}_{2n-1} - \bar{b}_{2n+1} + {}$$
$$+\frac{2}{\pi i}\sum_{k=1}^{N}\frac{4k}{4k^2-(2n-1)^2}\left[\frac{\varkappa-1}{2k} b_{2k} + \bar{b}_{2k+2} - \bar{b}_{2k}\right] + 2A_{2n-1}, \tag{23.21$''$}$$

$$\frac{1+\varkappa}{2n} b_{2n} = -\bar{b}_{2n+2} + \bar{b}_{2n} + \frac{4}{\pi i}\frac{\bar{b}_1}{4n^2-1} + {}$$
$$+\frac{2}{\pi i}\sum_{k=1}^{N}\frac{4k-2}{(2k-1)^2-4n^2}\left[\frac{\varkappa-1}{2k-1} b_{2k-1} + \bar{b}_{2k+1} - \bar{b}_{2k-1}\right] + 2A_{2n}$$
$$(n = 1, 2, \ldots, N) \tag{23.21$'''$}$$

and equations of the form

$$b_{m+2} - b_m = 0, \qquad m = 2N-1,\ 2N,\ \ldots . \tag{23.22}$$

Considering Eqs. (23.21) and (23.22) simultaneously gives a system of $2N+1$ linear equations in the first $2N+1$ unknowns $b_0, b_1, \ldots, b_{2N}$. The solution of this system will be taken as the required approximate solution to the original infinite system (23.20).

Note. The function $g(t)$ defined by equality (23.7) will not, for known reasons, be regular on the circumference even under the simplest loading of the boundary of the semi-circle. In the case of uniform normal pressure on γ^-, for example, the derivatives of $g(t)$ have discontinuities of the first kind at the points $t = \pm 1$. Consequently, the Fourier series for the function $g(t)$ is as a rule slowly convergent, and so one has to retain a large number of equations in the truncated system (23.21) to obtain a satisfactory numerical solution.

On the other hand, in all cases of greatest practical interest it is possible to "smooth" the function $g(t)$ beforehand by considering on γ^+ not an equality of the form of (23.6), but an equality resulting from it by transformation to the conjugate value. The function $g(t)$ then comes out smooth, but the infinite system of equations obtained in place of system (23.18) has a more complicated structure. Such a modification of the boundary condition is obviously equivalent to some transformation of system (23.18). A transformation of this kind may occasionally be found advantageous.

The boundary value problem (23.6) then takes the form

$$\varphi(t) + \varkappa\varphi(\bar{t}) + (t - \bar{t})\overline{\varphi'(t)} = g^*(t) + k(t)\Omega^*(t) \text{ on } \gamma, \tag{23.23}$$

$$g^*(t) = f(t) \text{ on } \gamma^-, \qquad g^*(t) = \bar{f}(t) \text{ on } \gamma^+, \tag{23.24}$$

$$\Omega^*(t) = \varphi(t) - \bar{\varphi}(t) + \varkappa[\varphi(\bar{t}) - \overline{\varphi(t)}] + (t - \bar{t})[\overline{\varphi'(t)} - \varphi'(\bar{t})]. \tag{23.25}$$

Instead of formulas (23.11) and (23.13), we have, respectively,

$$\Lambda_n^* = \frac{1}{2\pi i}\int_0^\pi \Omega^*(t)\, t^{-n-1}\, dt \qquad (n = 0, \pm 1, \ldots), \tag{23.26}$$

$$\Omega^*(t) = (\bar{a}_1 - a_1)\,t + \sum_{k=0}^{\infty}(a_k - \bar{a}_k)\,t^k - \sum_{k=0}^{\infty}\Omega_k^*[a; \varkappa]\,t^{-k}, \tag{23.27}$$

$$\Omega_k^*[a; \varkappa] = (k+2)(a_{k+2} - \bar{a}_{k+2}) - k(a_k - \bar{a}_k) - \varkappa(a_k - \bar{a}_k).$$

In the subsequent consideration we shall need the following obvious equalities obtainable by a termwise integration of the corresponding sums

$$\left.\begin{aligned}
&\frac{1}{\pi i}\int_0^\pi \Big[\sum_{k=0}^{\infty}\lambda_k t^k\Big] t^{-n-1}\,dt = \lambda_n + \frac{1}{\pi i}\sum_{k=0}^{\infty}{}' \,\delta_{k-n}\lambda_k \qquad (n \geqslant 0),\\
&\frac{1}{\pi i}\int_0^\pi \Big[\sum_{k=0}^{\infty}\lambda_k t^{-k}\Big] t^{-n-1}\,dt = \lambda_{-n} - \frac{1}{\pi i}\sum_{k=0}^{\infty}{}' \,\delta_{k+n}\lambda_k \qquad (n \leqslant 0),\\
&\qquad\qquad \frac{1}{\pi i}\int_0^\pi t^{-n}\,dt = \begin{cases} \dfrac{i}{2}\dfrac{\delta_{n-1}}{\pi} & (n \neq 1),\\ 1 & (n = 1),\end{cases}\\
&\qquad\qquad\qquad \delta_\nu = \frac{(-1)^\nu - 1}{\nu}.
\end{aligned}\right\} \tag{23.28}$$

On the basis of the above formulas it is now possible, similarly to the foregoing, to obtain an infinite system of linear equations for the determination of the unknowns a_k. In contrast to the previous discussion, here we need a complete system of equalities which are obtained by comparing the coefficients of all powers of t. This system, after elementary rearrangement, is written as

$$\left.\begin{aligned}
&\operatorname{Re}\{(1+\varkappa)a_0+2a_2\}=\\
&\qquad=\frac{1}{\pi}\operatorname{Im}\Big[\delta_1 a_1+\sum_{k=1}^{\infty}\delta_k a_k+\sum_{k=1}^{\infty}\delta_k\Omega_k^*[a;\varkappa]\Big]+A_0^*,\\
&\operatorname{Re}\{2a_1\}=\frac{1}{\pi}\operatorname{Im}\Big[{\sum_{k=1}^{\infty}}'\delta_{k-1}a_k+\sum_{k=0}^{\infty}\delta_{k+1}\Omega_k^*[a;\varkappa]\Big]+A_1^*,\\
&\operatorname{Re}\{(\varkappa-1)a_1+3a_3\}=\\
&\qquad=\frac{1}{\pi}\operatorname{Im}\Big[\sum_{k=0}^{\infty}\delta_{k+1}a_k+{\sum_{k=0}^{\infty}}'\delta_{k-1}\Omega_k^*[a;\varkappa]\Big]+A_{-1}^*,\\
&\operatorname{Re}\{a_n\}=\frac{1}{\pi}\operatorname{Im}\Big[\delta_{n-1}a_1+{\sum_{k=0}^{\infty}}'\delta_{k-n}a_k+\\
&\qquad\qquad+\sum_{k=0}^{\infty}\delta_{k+n}\Omega_k^*[a;\varkappa]\Big]+A_n^*\quad(n\geqslant 2),\\
&\operatorname{Re}\{(n+2)a_{n+2}-(n-\varkappa)a_n\}=\\
&=\frac{1}{\pi}\operatorname{Im}\Big[-\delta_{n+1}a_1+\sum_{k=0}^{\infty}\delta_{k+n}a_k+{\sum_{k=0}^{\infty}}'\delta_{k-n}\Omega_k^*[a;\varkappa]\Big]+A_{-n}^*.
\end{aligned}\right\}\quad(23.29)$$

In these equalities the A_n^* $(n = 0, \pm 1, \ldots)$ denote the Fourier coefficients of the function $g^*(t)$; the A_n^* are given real numbers. Equalities (23.29) represent an infinite system of real equations in the real and imaginary parts of a_k $(k = 0, 1, \ldots)$. Assume $a_k = \alpha_k' + i\beta_k'$ and make the substitution

$$\alpha_0' = \alpha_0, \qquad \beta_0' = \pi\beta_0, \qquad n\alpha_n' = \alpha_n, \qquad n\beta_n' = \pi\beta_n. \quad (23.30)$$

In terms of α_n, β_n system (23.29) then takes the form

$$\left.\begin{aligned}
&(1+\varkappa)\alpha_0+\alpha_2=\\
&\qquad=-2\beta_1+\sum_{k=1}^{\infty}\frac{\delta_k}{k}\beta_k+\sum_{k=1}^{\infty}\delta_k(\beta_{k+2}-\omega_k\beta_k)+A_0^*,\\
&2\alpha_1=\sum_{k=0}^{\infty}\frac{\delta_{k-1}}{k}\beta_k+{\sum_{k=0}^{\infty}}'\delta_{k+1}(\beta_{k+2}-\omega_k\beta_k)+\\
&\qquad\qquad+A_1^*\left(\omega_\nu=1+\frac{\varkappa}{\nu}\right),
\end{aligned}\right\}\quad(23.31)$$

$$\left.\begin{aligned}
&(\varkappa-1)\alpha_1+\alpha_3=\sum_{k=0}^{\infty}\frac{\delta_{k+1}}{k}\beta_k+\sum_{k=0}^{\infty}{}'\,\delta_{k-1}(\beta_{k+2}-\omega_k\beta_k)+A^*_{-1},\\
&\frac{\alpha_n}{n}=\delta_{n-1}\beta_1+\sum_{k=0}^{\infty}{}'\,\frac{\delta_{k-n}}{k}\beta_k+\sum_{k=0}^{\infty}\delta_{k+n}(\beta_{k+2}-\omega_k\beta_k)+A^*_n,\\
&\alpha_{n+2}-\omega_{-n}\alpha_n=-\delta_{n+1}\beta_1+\sum_{k=0}^{\infty}\frac{\delta_{k+n}}{k}\beta_k+\\
&\qquad\qquad+\sum_{k=0}^{\infty}{}'\,\delta_{k-n}(\beta_{k+2}-\omega_k\beta_k)+A^*_{-n}\quad(n\geqslant 2).
\end{aligned}\right\}\qquad(23.31)$$

By eliminating α_k $(k = 1, 2, \ldots)$ from this system, we obtain a system for the determination of β_k $(k = 0, 1, \ldots)$ which falls into two independent systems in the unknowns with even and odd subscripts. This system is of the form

$$\left.\begin{aligned}
&\omega_{01}\beta_0+\sum_{k=1}^{\infty}{}'\,\omega_{k1}\beta_k=B_1,\qquad \omega_{0n}\beta_0+\sum_{k=1}^{\infty}{}'\,\omega_{kn}\beta_k=B_n\quad(n\geqslant 2),\\
&\qquad\omega_{01}=-(\varkappa^2+4\varkappa+3),\qquad \omega_{0n}=(1+\varkappa)^2\delta_n,\\
&\omega_{k1}=((-1)^{k-1}-1)\times\\
&\qquad\times\left(\frac{4k+3\varkappa+3}{k(k+3)}+\frac{(1+\varkappa)^2(k-1)+k(5-8\varkappa-5k)}{2k(k^2-1)}\right),\\
&\omega_{kn}=((-1)^{k-n}-1)\times\\
&\qquad\times\left(\frac{(k-n)[(\varkappa+k)^2+1-kn-2k]-2\varkappa(k+n)}{k(k^2-n^2)}-\right.\\
&\qquad\qquad\left.-\frac{(k+n)[k-n-2(1-\varkappa)]}{(k+n)^2-4}\right)\quad(n=2, 3, \ldots),\\
&\qquad B_1={}^1/_2(\varkappa-1)A^*_1+3A^*_3-A^*_{-1},\\
&\qquad B_n=(n+2)A^*_{n+2}-(n-\varkappa)A^*_n-A^*_{-n}.
\end{aligned}\right\}\qquad(23.32)$$

After determining β_n, all α_n can be found directly from system (23.31) (α_0 is determined from the first equation of the system), whereupon the unknown holomorphic function $\varphi(z)$ is found.

Example. As a numerical example, consider the equilibrium of a semi-circle pressed against a rigid profile by a uniform pressure. Then

$$f(t)=pt\qquad(t \text{ on } \gamma^-),\qquad(23.33)$$

where p denotes the pressure intensity. For $g(t)$, according to formula (23.7), we have

$$g(t)=pe^{-i\vartheta}\quad(0\leqslant\vartheta\leqslant\pi),\qquad g(t)=pe^{i\vartheta}\qquad(\pi\leqslant\vartheta\leqslant 2\pi).$$

Consequently,

$$\operatorname{Re}\{g(t)\} = p\cos\vartheta, \qquad \operatorname{Im}\{g(t)\} = -p\,|\sin\vartheta\,|. \qquad (23.34)$$

The Fourier coefficients of the function $g(t)$ have the values

$$A_1 = \frac{p}{2}, \qquad A_{2n-1} = 0 \quad (n \geqslant 2), \qquad A_{2n} = \frac{2}{\pi i}\,\frac{p}{1-4n^2}. \qquad (23.35)$$

The truncated system (23.21), (23.22) consisting of $4N + 2$ real equations has been numerically solved for the values $N = 3, 4, 5$. For the computation under plane strain conditions we have assumed $\varkappa = 5/3$ (Poisson's ratio is taken equal to 1/3).

The boundary condition (23.5) having in this case (after substituting the corresponding series in the left side) the form

$$(1+\varkappa)\,b_0 + \bar{b}_2 + \bar{b}_1 e^{-i\vartheta} + \sum_{k=1}^{\infty} \frac{b_k}{k}\, e^{-ik\vartheta} +$$

$$+ \sum_{k=1}^{\infty} \left(\frac{\varkappa}{k}\, b_k + \bar{b}_{k+2} - \bar{b}_k\right) e^{ik\vartheta} = pe^{-i\vartheta} \qquad (0 \leqslant \vartheta \leqslant \pi) \qquad (23.36)$$

has been checked at the points $\vartheta = \vartheta_k = \frac{1}{2}\frac{\pi k}{N}$ $(k = 0, 1, \ldots, N)$. The real part of this equality is satisfied for the approximate solution thus found with great accuracy even with $N = 3$, and the discrepancy in its imaginary part, as might be expected, approaches zero rather slowly. We present the values of the normal pressure σ_y determined by formula (23.19) at the characteristic points of the punch base $(-1 \leqslant x \leqslant 1)$

$$\max \sigma_y = \sigma_y\,|_{x=0} = 1.1591p, \qquad \min \sigma_y = \sigma_y\,|_{x=\pm 1} = 0.8376\,p. \qquad (23.37)$$

The above approximate solution does not display the singular nature of the stress field near the angular points $z = \pm i$ which in our case are the points of changing boundary conditions. Nevertheless, it gives a fairly satisfactory picture of the state of stress in question as a whole. This is equally true for the solution indicated in the next section.

24. Sectionally Homogeneous Plane with a Circular Hole*

As another example of the application of the method, consider the problem of stresses in a sectionally homogeneous medium weakened by a symmetrical hole of circular shape.

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* See the author's paper [49].

Imagine a solid infinite plate composed of two materials with different elastic properties bonded along their common straight boundary. Let one of the component bodies with elastic characteristics, say, λ_1, μ_1 fill the lower half-plane of the plane of the variable $z = x + iy$ and the other, with characteristics λ_2, μ_2, the upper half-plane. Assume further that the sectionally homogeneous medium under consideration is weakened by a cut-out of circular shape and subjected to external forces applied at the lip and at infinity. The radius of the hole is taken to be unity and its centre is placed at the origin of co-ordinates. The interface of the media is then the real axis x with the removed segment $(-1, 1)$. This interface is denoted by L. Denote further by S^- and S^+ the lower and upper half-planes with the removed semi-circles, and by γ^- and γ^+, as before, the lower and upper semi-circumferences, respectively.

The unknown complex potentials φ and ψ which are holomorphic, respectively, in S^- and S^+ are assigned the same subscripts as the elastic constants. We have the following boundary value problem:

$$\left.\begin{aligned} &\varphi_1(t) + (t-\bar{t})\,\overline{\varphi_1'(t)} + \overline{\chi_1(t)} = f(t) \text{ on } \gamma^-,\\ &\varphi_2(t) + (t-\bar{t})\,\overline{\varphi_2'(t)} + \overline{\chi_2(t)} = f(t) \text{ on } \gamma^+, \end{aligned}\right\} \tag{24.1}$$

$$\left.\begin{aligned} &\varphi_1(t) + \overline{\chi_1(t)} = \varphi_2(t) + \overline{\chi_2(t)} \text{ on } L,\\ &\lambda\,[\varkappa_1\varphi_1(t) - \overline{\chi_1(t)}] = \varkappa_2\varphi_2(t) - \overline{\chi_2(t)} \text{ on } L. \end{aligned}\right\} \tag{24.2}$$

Here $f(t)$ is a given function of the point on the unit circumference,

$$\chi(z) = z\varphi'(z) + \psi(z), \qquad \lambda = \mu_2/\mu_1. \tag{24.3}$$

By adding and subtracting equalities (24.2) and transforming in the second one to conjugate values, we obtain on L

$$\left.\begin{aligned} &(1+\lambda\varkappa_1)\,\varphi_1(t) + (1-\lambda)\,\overline{\chi_1(t)} = (1+\varkappa_2)\,\varphi_2(t),\\ &(\varkappa_2+\lambda)\,\chi_1(t) + (\varkappa_2-\lambda\varkappa_1)\,\overline{\varphi_1(t)} = (1+\varkappa_2)\,\chi_2(t). \end{aligned}\right\} \tag{24.4}$$

From the above equalities it will be obvious that the functions φ_1, χ_1 defined in the region S^+ by the equalities $(z \in S^+)$

$$\left.\begin{aligned} &(1+\lambda\varkappa_1)\,\varphi_1(z) = (\lambda-1)\,\bar{\chi}_1(z) + (1+\varkappa_2)\,\varphi_2(z),\\ &(\varkappa_2+\lambda)\,\chi_1(z) = (\lambda\varkappa_1-\varkappa_2)\,\bar{\varphi}_1(z) + (1+\varkappa_2)\,\chi_2(z) \end{aligned}\right\} \tag{24.5}$$

analytically continue the values of the complex potentials φ_1, χ_1 to S^+ through the line L. In other words, the functions φ_1, χ_1 extended to S^+ by equalities (24.5) are, on the basis of formulas (24.4), holomorphic in the entire plane with the circular hole.

This extended region is denoted by S and the functions φ_1, χ_1 holomorphic in it by φ, χ.

The values of φ_2, χ_2 can now be expressed in terms of the newly introduced functions φ and χ.

From equalities (24.5), using the new notation, we have, for $z \in S^+$,

$$\left.\begin{aligned} (1+\varkappa_2)\,\varphi_2(z) &= (1+\lambda\varkappa_1)\,\varphi(z)+(1-\lambda)\,\overline{\chi}(z),\\ (1+\varkappa_2)\,\chi_2(z) &= (\varkappa_2+\lambda)\,\chi(z)+(\varkappa_2-\lambda\varkappa_1)\,\overline{\varphi}(z). \end{aligned}\right\} \qquad (24.6)$$

If φ_2, χ_2 defined by the above equalities are now substituted in the second condition of (24.1), then, for finding the functions φ, χ holomorphic in the region S, we obtain the following boundary conditions:

$$\left.\begin{aligned} &\varphi(t)+(t-\bar{t})\,\overline{\varphi'(t)}+\overline{\chi(t)}=f(t) \text{ on } \gamma^-,\\ &\varphi(t)+(t-\bar{t})\,\overline{\varphi'(t)}+\overline{\chi(t)}+\alpha\varphi(\bar{t})+\beta\overline{\chi(t)}+\delta\overline{\chi}(t)+\\ &\qquad\qquad +\delta(t-\bar{t})\,\chi'(\bar{t})=(1+\alpha)\,f(t) \text{ on } \gamma^+. \end{aligned}\right\} \qquad (24.7)$$

Here

$$\alpha=\frac{\mu_1\varkappa_2-\mu_2\varkappa_1}{\mu_1+\mu_2\varkappa_1}, \qquad \delta=\frac{\mu_1-\mu_2}{\mu_1+\mu_2\varkappa_1}, \qquad \beta=\alpha-\delta. \qquad (24.8)$$

To solve problem (24.7) we assume in the region S

$$\varphi(z)=\sum_{k=1}^{\infty} a_k z^{-k}, \qquad \chi(z)=\sum_{k=0}^{\infty} c_k z^{-k}. \qquad (24.9)$$

Supposing the uniform convergence on the circumference of the series obtained from expressions (24.9) by differentiation, we set up the left-hand sides of equalities (24.7). We have

$$\left.\begin{aligned} &\varphi(t)+(t-\bar{t})\,\overline{\varphi'(t)}+\overline{\chi(t)}=\bar{c}_0+\sum_{k=1}^{\infty} a_k t^{-k}+\sum_{k=1}^{\infty}\Omega'_k t^k,\\ &\alpha\varphi(\bar{t})+\beta\overline{\chi(t)}+\delta\overline{\chi}(t)+\delta(t-\bar{t})\,\chi'(\bar{t})=\\ &\qquad =\alpha\bar{c}_0+\sum_{k=1}^{\infty}\bar{c}_k t^{-k}+\sum_{k=1}^{\infty}\Omega''_k t^k \qquad (t=e^{i\vartheta},\ 0\leqslant\vartheta\leqslant 2\pi), \end{aligned}\right\} \qquad (24.10)$$

where

$$\left.\begin{aligned} \Omega'_k &= \bar{c}_k+k\bar{a}_k-(k-2)\,\bar{a}_{k-2},\\ \Omega''_k &= \alpha a_k+\beta\bar{c}_k+\delta\,[kc_k-(k-2)\,c_{k-2}] \qquad (k=1,2,\ldots). \end{aligned}\right\} \qquad (24.11)$$

In expressions (24.11) the term containing the factor $k-2$ must be omitted when $k=1$.

We further introduce a function $g(t)$ defined on the unit circumference as follows:

$$g(t)=f(t) \text{ on } \gamma^-, \qquad g(t)=(1+\alpha)\,f(t) \text{ on } \gamma^+. \qquad (24.12)$$

Assume

$$g(t)=\sum_{k=-\infty}^{\infty} A_k t^k,$$

$$A_k=\frac{1}{2\pi i}\int_0^{2\pi} g(t)\,t^{-k-1}\,dt \qquad (k=0,\ \pm 1,\ \pm 2,\ \ldots). \quad (24.13)$$

Substituting the above series in conditions (24.7) and comparing the coefficients of like powers of t^n ($n=0,\ \pm 1,\ \pm 2,\ \ldots$), we obtain, on the basis of equalities (23.28),

$$\left.\begin{aligned}
&\left(1+\frac{1}{2}\alpha\right)\bar{c}_0-\frac{\delta}{2\pi i}\sum_{k=1}^{\infty}\frac{(-1)^k-1}{k}\bar{c}_k+\\
&\qquad\qquad+\frac{1}{2\pi i}\sum_{k=1}^{\infty}\frac{(-1)^k-1}{k}\Omega_k''=A_0,\\
&\Omega_n'+\frac{1}{2}\Omega_n''+\frac{1}{2\pi i}\sum_{k=1}^{\infty}{}'\,\frac{(-1)^{k-n}-1}{k-n}\Omega_k''-\\
&\qquad-\frac{\delta}{2\pi i}\sum_{k=1}^{\infty}\frac{(-1)^{k+n}-1}{k+n}\bar{c}_k-\frac{\alpha}{2\pi i}\frac{(-1)^n-1}{n}\bar{c}_0=A_n,\\
&a_n+\frac{1}{2}\delta\bar{c}_n-\frac{\delta}{2\pi i}\sum_{k=1}^{\infty}{}'\,\frac{(-1)^{k-n}-1}{k-n}\bar{c}_k+\\
&\qquad+\frac{1}{2\pi i}\sum_{k=1}^{\infty}\frac{(-1)^{k+n}-1}{k+n}\Omega_k''+\frac{\alpha}{2\pi i}\frac{(-1)^n-1}{n}\bar{c}_0=A_{-n}\\
&\qquad\qquad\qquad(n=1,\ 2,\ 3,\ \ldots).
\end{aligned}\right\}\quad (24.14)$$

The set of equalities (24.14) represents an infinite system of linear algebraic equations in the unknown coefficients in expansions (24.9). For particular values of the elastic characteristics of the media, say for $\alpha=0$, when the elastic constants are related by the equation $\mu_1\varkappa_2=\mu_2\varkappa_1$, it is possible to eliminate all a_k ($k=1,\ 2,\ \ldots$) from equalities (24.14) by means of elementary operations and to obtain a system containing only the unknowns c_k ($k=0,\ 1,\ 2,\ \ldots$). In the general case, however, when the elastic characteristics are arbitrary, it is not advisable to strain after separation of the system.

For practical calculations one has to deal, as before, with a shortened system of linear equations which is obtained from equalities (24.14) by rejecting all equations and unknowns beginning with a certain number. To represent the unknowns we use the following uniform symbols:

$$x_{2n} = c_n \quad (n = 0, 1, \ldots), \quad x_{2n-1} = a_n \quad (n = 1, 2, \ldots) \tag{24.15}$$

and write the shortened system as

$$\left.\begin{aligned}
&\left(1+\frac{\alpha}{2}\right)\bar{x}_0 - \delta \sum_{k=1}^{N} \delta_k \bar{x}_{2k} + \sum_{k=1}^{N} \delta_k \Omega_{2k-1} = A_0, \\
&\Omega_{2n} + \frac{1}{2}\Omega_{2n-1} + \sum_{k=1}^{N}{}' \delta_{k-n}\Omega_{2k-1} - \delta \sum_{k=1}^{N} \delta_{k+n}\bar{x}_{2n} - \alpha\delta_n \bar{x}_0 = A_n, \\
&x_{2n-1} + \frac{\delta}{2}\bar{x}_{2n} - \delta \sum_{k=1}^{N}{}' \delta_{k-n}\bar{x}_{2k} + \sum_{k=1}^{N} \delta_{k+n}\Omega_{2k+1} + \alpha\delta_n \bar{x}_0 = A_{-n} \\
&\qquad\qquad (n = 1, \ldots, N),
\end{aligned}\right\} \tag{24.16}$$

where

$$\left.\begin{aligned}
&\delta_\nu = \frac{(-1)^\nu - 1}{2\pi i \nu} \qquad (\nu = 1, 2, \ldots), \\
&\Omega_{2k} = \bar{x}_{2k} + k\bar{x}_{2k-1} - (k-2)\bar{x}_{2k-5} \qquad (k = 1, 2, \ldots), \\
&\Omega_{2k-1} = \alpha x_{2k-1} + \beta\bar{x}_{2k} + \delta\,[k x_{2k} - (k-2) x_{2k-4}].
\end{aligned}\right\} \tag{24.17}$$

By solving system (24.14), we can determine both pairs of Kolosov-Muskhelishvili potentials by the above formulas, and consequently find all unknowns of our problem. In particular, the sum of the normal stresses on the contour of the hole, a quantity of practical interest, is given by the formula (it is assumed that no forces are applied at infinity)

$$\text{on } \gamma^- \qquad (\pi \leqslant \vartheta \leqslant 2\pi)$$

$$\sigma_r + \sigma_\vartheta = -4 \sum_{k=1}^{\infty} k\,[x'_{2k-1}\cos(k+1)\vartheta + x''_{2k-1}\sin(k+1)\vartheta]; \tag{24.18'}$$

$$\text{on } \gamma^+ \ (0 \leqslant \vartheta \leqslant \pi)$$

$$\sigma_r + \sigma_\vartheta = -4 \sum_{k=1}^{\infty} k\left[\left(\frac{x'_{2k-1}}{1+\alpha} + \eta x'_{2k}\right)\cos(k+1)\vartheta + \right.$$

$$\left. + \left(\frac{x''_{2k-1}}{1+\alpha} - \eta x''_{2k}\right)\sin(k+1)\vartheta\right] \tag{24.18''}$$

with

$$x_k = x'_k + i x''_k \qquad (k = 0, 1, \ldots), \qquad \eta = \frac{\mu_1 - \mu_2}{\mu_1(1+\varkappa_2)}.$$

Example. The computational scheme will be illustrated by the example of a non-homogeneous plane with a hole stretched at infinity by uniform forces in the direction of the x axis.

In the case under consideration we take approximately

$$f(t)=\frac{p}{2}(t^{-1}-t) \qquad (t=e^{i\vartheta}, \qquad 0\leqslant\vartheta\leqslant 2\pi), \qquad (24.19)$$

where p is a constant equal to the intensity of tensile force.*

For the Fourier coefficients of the function $g(t)$ defined by formula (24.12) we have

$$A_1=-\left(1+\frac{\alpha}{2}\right)\frac{p}{2}, \quad A_{-1}=\left(1+\frac{\alpha}{2}\right)\frac{p}{2}, \quad A_n=\frac{\alpha}{2\pi i}\frac{(-1)^{n-1}-1}{n^2-1}p$$
$$(n=0, \pm 2, \pm 3, \ldots). \qquad (24.20)$$

The numerical calculations will be carried out for equal Poisson's ratios when $\varkappa_1=\varkappa_2=\varkappa$. The foregoing constants α, β, ... may then be expressed in terms of $\varkappa$ and the ratio of the moduli of elasticity $\lambda=E_2/E_1$:

$$\alpha=\frac{\varkappa(1-\lambda)}{1+\lambda\varkappa}, \qquad \beta=\frac{(\varkappa-1)(1-\lambda)}{1+\lambda\varkappa}, \qquad \delta=\frac{1-\lambda}{1+\lambda\varkappa}, \qquad \eta=\frac{1-\lambda}{1+\varkappa}. \qquad (24.21)$$

The shortened system (24.16) has been programmed and solved using a БЭСМ-2 high-speed electronic computer for various values of the parameter λ when $\varkappa=2$ (Poisson's ratio has been taken equal to 0.25 for both media). As the calculations show, the boundary conditions of the problem (24.7) are already satisfied with reasonable accuracy when $N=8$, which amounts to considering a system of 34 real equations. We present the values of the stress concentration factors (k_1, k_2)

$$k=p^{-1}\max(\sigma_r+\sigma_\vartheta)$$

on the lower and upper semi-circumferences:

λ	0.25	0.50	0.75
k_1	3.5925	3.2854	3.1041
k_2	2.5885	2.7605	2.8952

It will be recalled that for the homogeneous case ($\lambda=1$) $k_1=k_2=3$.

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* Actually the quantity p is sectionally constant. If $\sigma_x^{(1,\infty)}$ is the tensile force in the lower half-plane, then $p=\sigma_x^{(1,\infty)}$ when t is on γ^-, $p=\lambda\sigma_x^{(1,\infty)}$ when t is on γ^+, where $\lambda=\mu_2/\mu_1$.

25. On One Problem of the Plane Theory of Elasticity

As a further application of the method developed in Sec. 22, consider the problem of the extension of an infinite plane weakened by a circular hole and an infinite straight-line cut starting at the lip of the hole and extending with the other end to infinity when the tensile forces of intensity, say, p are perpendicular to the line of the cut. Without loss of generality, the radius of the hole is taken to be unity and the cut is placed on the positive x axis from the point $x = 1$ to infinity (Fig. 1). The region occupied by the elastic medium is denoted by S and its boundary, a combination of the hloe contour and the cut with its two edges, by L. The boundary L is assumed to be free from external forces.

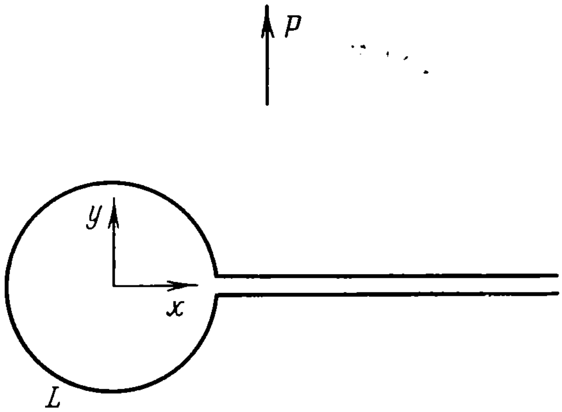

Fig. 1

We shall put the following content into the problem formulated above.

With our conditions concerning the external forces and the configuration of the region, we first assume the cut to be finite, viz. located on the segment of the real axis $1 \leqslant x \leqslant a$,* and consider the limiting state of elastic equilibrium when the end point of the cut $z = a$ moves into infinity along the x axis.

As the point $z = a$ recedes from the origin, the stresses in the neighbourhood of the point at infinity, of course, increase indefinitely. Suppose that these stresses have already been determined in some way and pose the problem of the elastic equilibrium of the plate corresponding to the given state of stress at infinity.

In determining the field in infinitely remote parts of the plane we use the condition that *it* (*i.e.*, *the field at infinity*) *should give no*

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* The case of a cut of finite length has been investigated in detail by L. A. Wigglesworth [146].

*external forces at the edges of the cut.** This condition is equivalent to requiring that the stresses in any finite part of the plane should be bounded. In other words, we shall study suitably introduced "finite parts" of the stresses arising near the hole in the elastic model represented in Fig. 1.

The unknown potentials of the problem, single-valued analytic (holomorphic) functions in S, are denoted by $\varphi_1(z)$, $\psi_1(z)$, $\Phi_1(z)$, $\Psi_1(z)$ and the boundary conditions of the problem are written as

$$\varphi_1(t)+t\overline{\varphi_1'(t)}+\overline{\psi_1(t)}=0 \text{ on } L. \tag{25.1}$$

With the use of the relation

$$z=\zeta^2 \qquad (\zeta=-\sqrt{z}), \tag{25.2}$$

where by $\sqrt{z}$ is meant the branch of the radical taking the positive values on the upper side of the cut, we map S onto the lower half-plane with the removed semi-circle. This region is designated Σ^-, the straight-line portions of its boundary are denoted by L_1 and L_2, and the curved portion of the boundary, i.e., the semi-circumference $|\sigma|=1$, $\operatorname{Im}\sigma<0$, by γ^-. The region symmetrical to Σ^- about the real axis in the ζ plane is denoted by Σ^+ and the curved portion of its boundary by γ^+.

With the additional notation

$$\varphi_1(z)=\varphi_0(\zeta), \qquad \psi_1(z)=\psi_0(\zeta), \qquad \Phi_1(z)=\Phi_0(\zeta)=\frac{\varphi_0'(\zeta)}{2\zeta} \tag{25.3}$$

condition (25.1) transforms to

$$\varphi_0(\sigma)+\frac{\sigma}{2}\overline{\varphi_0'(\sigma)}+\overline{\psi_0(\sigma)}=0 \text{ on } L_1+L_2, \tag{25.4}$$

$$\varphi_0(\sigma)+\frac{\sigma^3}{2}\overline{\varphi_0'(\sigma)}+\overline{\psi_0(\sigma)}=0 \text{ on } \gamma^-. \tag{25.5}$$

We introduce a holomorphic function $\varphi_0(\zeta)$ in the region Σ^+ by the equality

$$\varphi_0(\zeta)=-\frac{\zeta}{2}\overline{\varphi}_0'(\zeta)-\overline{\psi}_0(\zeta) \qquad (\zeta \text{ in } \Sigma^+) \tag{25.6}$$

and define $\psi_0(\zeta)$ in terms of the extended function $\varphi_0(\zeta)$ as

$$\psi_0(\zeta)=-\overline{\varphi}_0(\zeta)-\frac{\zeta}{2}\varphi_0'(\zeta) \qquad (\zeta \text{ in } \Sigma^-). \tag{25.7}$$

Then the boundary condition (25.4) is fulfilled because of the analyticity of the function $\varphi_0(\zeta)$ outside the circumference γ, $\gamma=\gamma^-+ +\gamma^+$, and condition (25.5) becomes, by virtue of (25.7),

$$\varphi_0(\sigma)-\varphi_0(\overline{\sigma})+(\sigma^2-\overline{\sigma}^2)\overline{\Phi_0(\sigma)}=0 \quad \text{on } \gamma^-. \tag{25.8}$$

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[*] The problem under consideration for the case of external forces not affecting the tip of the cut ($z=\infty$) has been solved by the same method as in Sec. 22 by G. A. Kutateladze [68]. In this case the stresses are bounded everywhere and the problem is solvable in the usual formulation.

Thus, we have arrived at problem (25.8) for determining the single function $\varphi_0(\zeta)$ holomorphic for $|\zeta| > 1$.

In order to establish the form of the function $\varphi_0(\zeta)$ in the neighbourhood of the point $\zeta = \infty$, we resort to the following procedure.

Consider the problem of the extension of a plane cut along the segment $0 \leqslant x \leqslant a$ of the real axis x when the edges of the cut are free from forces and the tensile forces p at infinity are perpendicular to the line of the cut. The complex potential $\Phi_*(\zeta)$ corresponding to this problem is of the form (N. I. Muskhelishvili [93], Sec. 120)

$$\Phi_*(z) = \frac{c_0 z + c_1}{\sqrt{z(z-a)}} - \frac{p}{4}, \tag{25.9}$$

where

$$c_0 = \frac{p}{2}, \qquad \int_0^a \frac{(c_0 t + c_1)\,dt}{\sqrt{t(t-a)}} = 0.$$

By $\sqrt{z(z-a)}$ we understand the branch of the radical having, for large $|z|$, the form

$$\sqrt{z(z-a)} = z + O(1)$$

and by $\sqrt{t(t-a)}$ for $0 < t < a$ the values of this function on the upper side of the cut. Then

$$\int_0^a \frac{dt}{\sqrt{t(t-a)}} = -\pi i, \qquad \int_0^a \frac{t\,dt}{\sqrt{t(t-a)}} = -\frac{a}{2}\pi i.$$

These equalities determine the constant c_1 in formula (25.9) as

$$c_1 = -\frac{ap}{4}.$$

Consequently,

$$\Phi_*(z) = \frac{p}{2}\frac{z}{\sqrt{z(z-a)}} - \frac{p}{4}\frac{a}{\sqrt{z(z-a)}} - \frac{p}{4}. \tag{25.10}$$

Consider now the function $\Phi_0(z)$ in the neighbourhood of the right end of the cut $z = a$

$$z - a = -h, \qquad |h| \leqslant \rho, \tag{25.11}$$

where ρ is a sufficiently small positive number. Noting that

$$\frac{a}{\sqrt{z(z-a)}} = \frac{1}{i}\frac{a}{\sqrt{z(a-z)}} = \frac{1}{i}\left\{\sqrt{\frac{z}{a-z}} + \sqrt{\frac{a-z}{z}}\right\},$$

we represent formula (25.10) as

$$\Phi_*(z) = \frac{p}{4i}\sqrt{\frac{z}{a-z}} - \frac{p}{4i}\sqrt{\frac{a-z}{z}} - \frac{p}{4}. \tag{25.12}$$

As the point $x = a$ moves indefinitely far away from the origin and if at the same time condition (25.11) is fulfilled for a fixed ρ, the right-hand side of expression (25.12) behaves as a function of the complex argument w:

$$\frac{p}{4i}\left[\sqrt{w}-\frac{1}{\sqrt{w}}\right]-\frac{p}{4} \quad \text{as} \quad |w|\to\infty.$$

From this we conclude that the complex potential $\Phi_1(z)$ of our problem admits, for large $|z|$, the representation

$$\Phi_1(z)=\frac{p}{4i}\sqrt{z}-\frac{p}{4}-\frac{p}{4i}\frac{1}{\sqrt{z}}+o\left(\frac{1}{\sqrt{z}}\right). \qquad (25.13)$$

On the basis of formula (25.13), if use is made of the relation

$$\Phi_1(z)=\Phi_0(\zeta)=\frac{\varphi_0'(\zeta)}{2\zeta},$$

we obtain for $\varphi_0(\zeta)$, $\Phi_0(\zeta)$ representations suitable in infinitely remote parts of the ζ plane

$$\varphi_0(\zeta)=p_0(\zeta)+\varphi(\zeta), \qquad (25.14)$$

$$\Phi_0(\zeta)=P_0(\zeta)+\Phi(\zeta), \qquad (25.15)$$

$$p_0(\zeta)=-\frac{p}{2i}\left(\frac{\zeta^3}{3}+\frac{i\zeta^2}{2}-\zeta\right), \quad P_0(\zeta)=-\frac{p}{4i}\left(\zeta+i-\frac{1}{\zeta}\right), \qquad (25.16)$$

$$\Phi(\zeta)=\frac{\varphi'(\zeta)}{2\zeta},$$

where $\varphi(\zeta)$ is a holomorphic function outside the circumference γ, including the point at infinity. It must be determined from the boundary condition (25.8).*

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* The choice of the principal part of the function $\psi_0(\zeta)$ is outside our control. This principal part, as the function ψ_0 itself, is completely determined by the possibility for the analytic continuation through L_1+L_2 of the potential $\varphi_0(\zeta)$ given in the region Σ^+ by equality (25.6). Indeed, from formulas (25.7) and (25.14) we find, for large $|\zeta|$,

$$\psi_0(\zeta)=\frac{p}{4i}\left(\frac{\zeta^3}{3}+2i\zeta^2+\zeta\right)+\psi(\zeta) \quad (\text{in } \Sigma^-), \qquad (a)$$

where $\psi(\zeta)$ is a bounded function

$$\psi(\zeta)=-\bar{\varphi}(\zeta)-\frac{\zeta}{2}\varphi'(\zeta)=O\left(\frac{1}{\zeta}\right) \quad (\text{in } \Sigma^-).$$

From (a) we have, for large $|z|$,

$$\Psi_1(z)=\psi_1'(z)=-\frac{p}{8i}\left(\sqrt{z}-4i+\frac{1}{\sqrt{z}}\right)+O\left(\frac{1}{\sqrt{z}}\right). \qquad (b)$$

It can easily be verified that for a state of stress in the plate characterized by the first three terms on the right-hand sides of formulas (25.13) and (b) no external forces will arise at the edges of the cut.

For the function $\varphi(\zeta)$ we use the power series expansion

$$\varphi(\zeta) = \sum_{k=1}^{\infty} a_k \zeta^{-k}. \tag{25.17}$$

Accordingly,

$$\Phi(\zeta) = -\frac{1}{2} \sum_{k=1}^{\infty} k a_k \zeta^{-k-2}. \tag{25.18}$$

Inserting the function $\varphi_0(\zeta)$ from expression (25.14) in relation (25.8), we obtain

$$\varphi(\sigma) - \varphi(\bar{\sigma}) + (\sigma^2 - \bar{\sigma}^2)\,\overline{\Phi(\sigma)} = f(\sigma) \quad \text{on } \gamma^-, \qquad (25.19$$

$$f(\sigma) = -\{p_0(\sigma) - p_0(\bar{\sigma}) + (\sigma^2 - \bar{\sigma}^2)\,\overline{P_0(\sigma)}\} \quad \text{on } \gamma^-, \tag{25.20}$$

where $p_0(\sigma)$, $P_0(\sigma)$ are defined by formulas (25.16).

We extend equality (25.19) to the upper semi-circumference γ^+ and write the boundary conditions of our problem in the following final form:

$$\varphi(\sigma) - \varphi(\bar{\sigma}) + \Omega(\sigma) = F(\sigma) \quad \text{on } \gamma, \tag{25.21}$$

where

$$\Omega(\sigma) = \begin{cases} (\sigma^2 - \bar{\sigma}^2)\,\overline{\Phi(\sigma)} & \text{on } \gamma^-, \\ (\sigma^2 - \bar{\sigma}^2)\,\overline{\Phi(\bar{\sigma})} & \text{on } \gamma^+, \end{cases} \qquad F(\sigma) = \begin{cases} f(\sigma) & \text{on } \gamma^-, \\ -f(\bar{\sigma}) & \text{on } \gamma^+. \end{cases} \tag{25.22}$$

Assume further

$$\Omega(\sigma) = \sum_{-\infty}^{\infty} \Lambda_k \sigma^k, \qquad F(\sigma) = \sum_{-\infty}^{\infty} A_k \sigma^k. \tag{25.23}$$

According to the definition of the function $\Omega(\sigma)$ we have

$$\Lambda_n = \frac{1}{2\pi i} \int_{\gamma^-} (\sigma^2 - \bar{\sigma}^2)\,\overline{\Phi(\sigma)}\,\sigma^{-n-1}\,d\sigma + {}$$

$$+ \frac{1}{2\pi i} \int_{\gamma^+} (\sigma^2 - \bar{\sigma}^2)\,\overline{\Phi(\bar{\sigma})}\,\sigma^{-n-1}\,d\sigma = \Lambda_-^{(n)} + \Lambda_+^{(n)}. \tag{25.24}$$

By means of the substitution $\sigma = \frac{1}{\sigma'}$ we transform the second integral to

$$\frac{1}{2\pi i} \int_{\gamma^+} (\sigma^2 - \bar{\sigma}^2)\,\overline{\Phi(\bar{\sigma})}\,\sigma^{-n-1}\,d\sigma =$$

$$= -\frac{1}{2\pi i} \int_{\gamma^-} (\sigma^2 - \bar{\sigma}^2)\,\overline{\Phi(\sigma)}\,\sigma^{n-1}\,d\sigma = -\Lambda_-^{(-n)}.$$

Consequently,

$$\Lambda_n = \Lambda_-^{(n)} - \Lambda_-^{(-n)}. \tag{25.25}$$

Similarly, we find

$$A_n = A_-^{(n)} - A_-^{(-n)}. \tag{25.26}$$

From the above equalities it follows that $\Lambda_{-n} = -\Lambda_n$, $A_{-n} = -A_n$, and so we may restrict our consideration to the Fourier coefficients Λ_n, A_n with positive subscripts.

According to formula (25.18) we have

$$(\sigma^2 - \bar{\sigma}^2)\,\overline{\Phi(\sigma)} = -\frac{1}{2}\sum_{k=1}^{\infty} k\bar{a}_k\,(\sigma^{k+4} - \sigma^k).$$

From this we find after elementary computations

$$\left.\begin{aligned} \Lambda_-^{(n)} &= \frac{1}{2\pi i}\int_{\gamma^-} (\sigma^2 - \bar{\sigma}^2)\,\overline{\Phi(\sigma)}\,\sigma^{-n-1}\,d\sigma = \\ &= \frac{1}{4}\,[n\bar{a}_n - (n-4)\,\bar{a}_{n-4}] - \frac{1}{\pi i}\sum_{k=1}^{\infty}{}'' \frac{k\bar{a}_k\delta_{k-n}}{k-n+4}, \\ \Lambda_-^{(-n)} &= -\frac{1}{\pi i}\sum_{k=1}^{\infty}\frac{k\bar{a}_k\delta_{k+n}}{k+n+4} \qquad (n \geqslant 1), \end{aligned}\right\} \tag{25.27}$$

where

$$\delta_m = \frac{(-1)^m - 1}{m}, \quad m \neq 0, \quad \delta_0 = -\pi i$$

and the symbol Σ'' denotes that the summation leaves out two values of the subscript k, viz. $k = n$, $k = n - 4$.

The above formulas, with (25.25), give

$$\begin{aligned} \Lambda_n &= \frac{1}{4}\,[n\bar{a}_n - (n-4)\,\bar{a}_{n-4}] + \\ &\quad + \frac{1}{\pi i}\sum_{k=1}^{\infty} k\bar{a}_k\left[\frac{\delta_{k+n}}{k+n+4} - \frac{\delta_{k-n}}{k-n+4}\right] \qquad (n \geqslant 1). \end{aligned} \tag{25.28}$$

By using the second of formulas (25.22) and taking (25.20), (25.16) into account, for the Fourier coefficients of the function $F(\sigma)$ we find in a similar way

$$\left.\begin{aligned} &A_1 = \frac{pi}{2}, \quad A_3 = -\frac{pi}{6}, \quad A_2 = \frac{p\lambda_2}{4\pi} + \frac{p}{2}, \\ &A_n = \frac{p\lambda_n}{4\pi} \quad (n \geqslant 4), \qquad \lambda_n = \delta_{3-n} - \sigma_{3+n} + \delta_{n+1} + \delta_{n-1}. \end{aligned}\right\} \tag{25.29}$$

Upon substitution of the corresponding series in equality (25.21) the latter becomes

$$\sum_{k=1}^{\infty} a_k\,(\bar{\sigma}^k - \sigma^k) + \sum_{-\infty}^{\infty} \Lambda_k\sigma^k = \sum_{-\infty}^{\infty} A_k\sigma^k, \tag{25.30}$$

where the coefficients Λ_n and A_n for $n \geqslant 1$ are defined by formulas (25.28) and (25.29).

Comparison of the coefficients of σ^n $(n \geqslant 1)$ in equality (25.30) gives the system of equations

$$-a_n + \Lambda_n = A_n, \qquad n \geqslant 1. \tag{25.31}$$

By the definition of the symbol δ_m, for the numbers λ_n we find after elementary computations

$$\lambda_n = \frac{[1-(-1)^{n-1}]\,16n}{(n^2-1)(n^2-9)}, \qquad n \neq 1,\ 3 \tag{25.32}$$

from which it follows that all coefficients A_n with odd subscripts, except A_1 and A_3, are zero: $A_{2n-1} = 0$, $n \geqslant 3$.

We separate real from imaginary parts in equalities (25.31), take into account formulas (25.29), (25.32) and introduce the notation

$$na_n = x_n + iy_n \qquad (n \geqslant 1). \tag{25.33}$$

We then obtain an infinite system of linear algebraic equations in the real unknowns x_n, y_n:

$$\left.\begin{aligned}
&-5y_1 = \frac{4}{\pi}\sum_{k=1}^{\infty} x_{2k}\left(\frac{\delta_{2k+1}}{2k+5} - \frac{\delta_{2k-1}}{2k+3}\right) + 2p,\\
&-\left(1+\frac{4}{3}\right) y_3 = \frac{4}{\pi}\sum_{k=1}^{\infty} x_{2k}\left(\frac{\delta_{2k+3}}{2k+7} - \frac{\delta_{2k-3}}{2k+1}\right) - \frac{2}{3}p,\\
&-\left(1+\frac{4}{2n-1}\right) y_{2n-1} + y_{2n-5} =\\
&\qquad = \frac{4}{\pi}\sum_{k=1}^{\infty} x_{2k}\left(\frac{\delta_{2k+2n-1}}{2k+2n+3} - \frac{\delta_{2k-2n+1}}{2k-2n+5}\right) \quad (n \geqslant 3),\\
&-x_2 = \frac{4}{\pi}\sum_{k=1}^{\infty} y_{2k-1}\left(\frac{\delta_{2k+1}}{2k+5} - \frac{\delta_{2k-3}}{2k+1}\right) + 4A_2,\\
&0 = \frac{4}{\pi}\sum_{k=1}^{\infty} y_{2k-1}\left(\frac{\delta_{2k+3}}{2k+7} - \frac{\delta_{2k-5}}{2k-1}\right) + 4A_4,\\
&\left(1-\frac{2}{n}\right) x_{2n} - x_{2n-4} =\\
&= \frac{4}{\pi}\sum_{k=1}^{\infty} y_{2k-1}\left(\frac{\delta_{2k+2n-1}}{2k+2n+3} - \frac{\delta_{2k-2n-1}}{2k-2n+3}\right) + 4A_{2n} \quad (n \geqslant 3),
\end{aligned}\right\} \tag{25.34}$$

$$x_{2n-1} = y_{2n} = 0 \quad n \geqslant 1, \tag{25.34'}$$

$$A_2 = -\frac{16}{15}\frac{p}{\pi} + \frac{p}{2}, \qquad A_{2n} = \frac{16n}{(4n^2-1)(4n^2-9)}\frac{p}{\pi}, \quad n \geqslant 2. \tag{25.35}$$

After solving system (25.34), we determine the function $\Phi(\zeta)$, whereupon all unknowns of our problem can be found without much difficulty. In particular, the sum of the normal stresses is defined by the formula

$$\sigma_\vartheta + \sigma_\rho = 2\left[\Phi_0(\zeta) + \overline{\Phi_0(\zeta)}\right] =$$
$$= -4\mathrm{Re}\left\{\frac{p}{4i}\left(\zeta + i - \frac{1}{\zeta}\right) + \frac{\zeta^{-2}}{2}\sum_{k=1}^{\infty}(x_k + iy_k)\zeta^{-k}\right\}, \quad \zeta = \rho e^{i\vartheta} \tag{25.36}$$

On the circumference $|\zeta| = 1$ we have: $\sigma_\rho = 0$; the above formula, with (25.34′), becomes:

$$-\sigma_\vartheta = 1 + 2\sin\vartheta + 2\sum_{k=1}^{\infty}\left[x_{2k}\cos(2k+2)\vartheta + y_{2k-1}\sin(2k+1)\vartheta\right]$$
$$(\pi \leqslant \vartheta \leqslant 2\pi). \tag{25.37}$$

In practical calculations we shall use a finite system of linear equations which is obtainable from system (25.34) by the usual shortening procedure. This system of $2N$ equations (N is some natural number) with the same number of unknowns $y_1, y_3, \ldots, y_{2N-1}$, $x_2, x_4, \ldots, x_{2N}$ is of the form

$$\left.\begin{aligned}
&5y_1 = \sum_{k=1}^{N}\alpha_{k1}x_{2k} - 2p,\\
&\frac{7}{3}y_3 = \sum_{k=1}^{N}\alpha_{k2}x_{2k} + \frac{2}{3}p,\\
&\left(1 + \frac{4}{2n-1}\right)y_{2n-1} - y_{2n-5} = \sum_{k=1}^{N}\alpha_{kn}x_{2k} \quad (n = 3, 4, \ldots, N),\\
&-x_2 = \sum_{k=1}^{N}\beta_{k1}y_{2k-1} + A_2' + 2p,\\
&0 = \sum_{k=1}^{N}\beta_{k2}y_{2k-1} + A_4',\\
&\left(1 - \frac{2}{n}\right)x_{2n} - x_{2n-4} = \sum_{k=1}^{N}\beta_{kn}y_{2k-1} + A_{2n}' \quad (n = 3, 4, \ldots, N),
\end{aligned}\right\} \tag{25.38}$$

where

$$\alpha_{kn}=-\frac{64}{\pi}\frac{(2n-1)(k+1)}{(2k+2n-1)(2k+2n+3)(2k-2n+1)(2k-2n+5)},$$

$$\beta_{kn}=\frac{64}{\pi}\frac{n(2k+1)}{(2k+2n-1)(2k+2n+3)(2k-2n-1)(2k-2n+3)},$$

$$A'_{2n}=\frac{64}{\pi}\frac{np}{(4n^2-1)(4n^2-9)}.$$

The calculations show that, in order to satisfy the boundary conditions of the problem with reasonable accuracy, we may limit system (25.38) to 20 equations ($N=10$). System (25.38) has been solved with $N=5$, 10, 15 and 20 using an electronic computer, and the corresponding values of the tensile stress σ_ϑ at the points of the contour of the hole have been found by formula (25.37). The results of the calculations for $N=20$ are given below in tabular form.

Stress Values

ϑ	π	$\frac{21}{20}\pi$	$\frac{22}{20}\pi$	$\frac{23}{20}\pi$	$\frac{24}{20}\pi$	$\frac{25}{20}\pi$	$\frac{26}{20}\pi$	$\frac{27}{20}\pi$	$\frac{28}{20}\pi$	$\frac{29}{20}\pi$	$\frac{3}{2}\pi$
σ_ϑ/p	–0.08	–0.16	–0.88	–1.62	–1.91	–1.54	–0.61	0.64	1.85	2.72	3.03

As might be expected, the maximum contour stress is attained at the point $z=-1$ and is numerically equal to $3.03p$. This value differs only slightly from the maximum of the same stress σ_ϑ on the (force-free) contour of a circular hole in a plate with no cut under uniaxial tension.

Note. The formulated problem without the additional condition is incorrect. On the other hand, the condition of no external forces at the edges of the cut means, in fact, the rejection of the cut, and so, physically, the modified problem does not differ from the problem of the extension of a plane with a circular hole. A good agreement of the two solutions illustrates the effectiveness of the method used, which is the main object of the present section.

26. Application to the Torsion Problem for Elastic Bars*

In the torsion problem involving the determination of a single function of a complex argument holomorphic in the particular region under consideration, the foregoing method, as will presently be shown, furnishes a rather elementary means of finding a solution in closed form for a certain class of regions.

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* The results discussed in Secs. 26 and 27 have been obtained by G. A. Kutateladze [64 to 67].

1°. General solution of the torsion problem. We shall consider an elastic prismatic bar of homogeneous and isotropic material with a uniform cross section. It is assumed that the lateral surface of the bar is free from external loads, one of the bases is fixed, and at the other are applied forces statically equivalent to a twisting couple whose moment vector is perpendicular to the plane of the base.

Let the physical region S of any one section of the bar be placed in the plane of the variable z, and, along with it, introduce another plane of an auxiliary variable $\zeta = \xi + i\eta$. Consider in the ζ plane a concentric circular ring with centre at the origin. Let the outer boundary of the ring be γ_1, and the inner boundary γ_2. Let, further, ρ_1 and ρ_2 denote the radii of the corresponding circles, l the set of two segments of the real axis lying within the ring, Σ^+ and Σ^- the upper and lower half-rings, and γ_1^+, γ_2^+ and γ_1^-, γ_2^- the upper and lower semi-circumferences, respectively.

According to the condition adopted in Sec. 22 we suppose that the function

$$z = \omega(\zeta), \tag{26.1}$$

which maps conformally the region S onto the half-ring Σ^+, admits analytic continuation into the lower half-ring Σ^- through the line l.

The torsion problem for the bar under consideration is then reduced to finding a function of a complex variable $F(\zeta)$ holomorphic in the region Σ^+ from the boundary condition (N. I. Muskhelishvili [93], Sec. 134)

$$F(\sigma) + \overline{F(\sigma)} = \omega(\sigma)\,\overline{\omega(\sigma)} \text{ on } \gamma \quad (\gamma = \gamma_1^+ + \gamma_2^+ + l), \tag{26.2}$$

where σ denotes the value of the argument $\zeta = \rho e^{i\vartheta}$ on γ and $F(\sigma)$ the boundary values of the function $F(\zeta)$.

The function $F(\zeta)$ is related to the so-called complex torsion function $f(\zeta)$ expressed in terms of the variable ζ by the equation

$$f(\zeta) = iF(\zeta). \tag{26.3}$$

We extend the definition of the function $F(\zeta)$ to the lower half-ring Σ^- by the equality

$$F(\zeta) = -\bar{F}(\zeta) + \omega(\zeta)\,\bar{\omega}(\zeta) \qquad (\zeta \in \Sigma^-). \tag{26.4}$$

It is easily verified that by virtue of the boundary condition (26.2) on the line l the function $F(\zeta)$ can be continued analytically from the upper into the lower half-ring through l.

We replace the argument ζ by $\bar{\zeta}$ in expression (26.4), assuming that $\zeta \in \Sigma^+$. We obtain

$$\overline{F(\zeta)} = -F(\bar{\zeta}) + \omega(\bar{\zeta})\,\overline{\omega(\zeta)} \qquad (\zeta \in \Sigma^+).$$

On the basis of the last equality condition (26.2) on γ_1^+ and γ_2^+ becomes

$$F(\sigma) - F(\bar{\sigma}) = \overline{\omega(\sigma)}\,[\omega(\sigma) - \omega(\bar{\sigma})] \quad \text{on} \quad \gamma_1^+ + \gamma_2^+. \tag{26.5}$$

The boundary conditions on the lower semi-circumferences are obtained from the above equality if σ is replaced by $\bar{\sigma}$ in it, assuming the point σ to belong to the arcs γ_j^-. We have

$$F(\sigma) - F(\bar{\sigma}) = \bar{\omega}(\sigma)\,[\omega(\sigma) - \omega(\bar{\sigma})] \quad \text{on} \quad \gamma_1^- + \gamma_2^-. \tag{26.6}$$

Consequently, the solution of the problem under consideration reduces to determining the function $F(\zeta)$ holomorphic in the entire concentric ring $\Sigma = \Sigma^+ + \Sigma^- + l$ from the following boundary conditions:

$$F(\sigma) - F(\bar{\sigma}) = \begin{cases} \Omega'(\sigma) & \text{on} \quad \gamma_1^+ + \gamma_2^+, \\ \Omega''(\sigma) & \text{on} \quad \gamma_1^- + \gamma_2^-, \end{cases} \tag{26.7}$$

where $\Omega'(\sigma)$ and $\Omega''(\sigma)$ stand for the right-hand sides of equalities (26.5) and (26.6), respectively.

By using the power series method for the solution of the mixed problem (26.7), we assume in the ring

$$F(\zeta) = \sum_{k=1}^{\infty} (a_k \zeta^k + b_k \zeta^{-k}). \tag{26.8}$$

After elementary computations the unknown coefficients a_k, b_k are obtained as

$$a_k = \frac{A_k^{(1)} \rho_1^k - A_k^{(2)} \rho_2^k}{\rho_1^{2k} - \rho_2^{2k}}, \qquad b_k = \rho_1^k \rho_2^k \frac{A_k^{(1)} \rho_2^k - A_k^{(2)} \rho_1^k}{\rho_1^{2k} - \rho_2^{2k}}, \tag{26.9}$$

where the $A_k^{(j)}$ $(j = 1, 2)$ represent the coefficients in the Fourier series of the given functions on the boundary of the ring

$$A_k^{(j)} = -A_{-k}^{(j)} = \frac{1}{2\pi} \int\limits_{\gamma_j^+} \Omega'(\sigma)\, \sigma^{-(k+1)}\, d\sigma + \frac{1}{2\pi} \int\limits_{\gamma_j^-} \Omega''(\sigma)\, \sigma^{-(k+1)}\, d\sigma. \tag{26.10}$$

The problem is thus solved. For the analysis of the elastic equilibrium of a twisted bar it remains to use the available well-known formulas (N. I. Muskhelishvili [93], Sec. 134).

For example, the stress field at an arbitrary point of the bar section is determined by the formula

$$T_\rho - iT_\vartheta = \frac{\tau\mu\zeta}{\rho\,|\omega'(\zeta)|} \{f'(\zeta) - i\omega'(\zeta)\,\overline{\omega(\zeta)}\}, \tag{26.11}$$

where μ is the shearing modulus, τ is the amount of twist, T_ρ and T_ϑ are the components of the shearing stress vector in curvilinear coordinates.

The torsional rigidity is

$$D = \mu I + \mu D_0, \tag{26.12}$$

where

$$I = \frac{1}{4i}\int\limits_{\gamma} \overline{\omega^2(\sigma)}\,\omega(\sigma)\,d\omega(\sigma), \qquad D_0 = -\frac{1}{4}\int\limits_{\gamma} [f(\sigma)+\overline{f(\sigma)}]\,d\{\omega(\sigma)\overline{\omega(\sigma)}\},$$

remembering that γ denotes the complete boundary of the half-ring, $\gamma = \gamma_1^+ + \gamma_2^+ + l$.

Consider, separately, the case when we know a function $\omega(\zeta)$ which maps the region of the cross section onto the upper semi-circle of unit radius in the ζ plane and which admits, as before, analytic continuation into the lower semi-circle through its diameter l. In the previous notation, we have, in accordance with conditions (26.7), the boundary problem

$$F(\sigma) - F(\bar{\sigma}) = \begin{cases} \overline{\omega(\sigma)}\,[\omega(\sigma) - \omega(\bar{\sigma})] & \text{on } \gamma^+, \\ \bar{\omega}(\sigma)\,[\omega(\sigma) - \omega(\bar{\sigma})] & \text{on } \gamma^-. \end{cases} \tag{26.13}$$

The solution of this problem is represented as

$$F(\zeta) = \sum_{k=1}^{\infty} A_k^{(1)}\zeta^k,$$

where the coefficients $A_k^{(1)}$ are determined from relation (26.10) with $\rho_1 = 1$ $(\rho_2 = 0)$.

On the other hand, the holomorphic function $F(\zeta)$ in the circle, whose real part is given on the circumference $|\sigma| = 1$ by virtue of formula (26.13), can be expressed according to the well-known Schwarz integral

$$F(\zeta) = \frac{1}{2\pi i}\int\limits_{\gamma} [F(\sigma) + \overline{F(\sigma)}]\,\frac{d\sigma}{\sigma - \zeta} + \text{constant}$$

The above formula, as applied to problem (26.13), gives

$$F(\zeta) = \frac{1}{2\pi i}\int\limits_{\gamma^+} \overline{\omega(\sigma)}\,[\omega(\sigma) - \omega(\bar{\sigma})]\left[\frac{1}{\sigma - \zeta} + \frac{\zeta}{\sigma\zeta - 1} - \frac{1}{\sigma}\right] d\sigma + \text{constant}.$$

Consequently, the complex torsion function in our case can be expressed by both the power series

$$f(\zeta) = i\sum_{k=1}^{\infty} A_k\zeta^k \qquad (A_k = A_k^{(1)})$$

and quadrature

$$f(\zeta) = \frac{1}{2\pi}\int\limits_{\gamma^+} \overline{\omega(\sigma)}\,[\omega(\sigma) - \omega(\bar{\sigma})]\left[\frac{1}{\sigma - \zeta} + \frac{\zeta}{\sigma\zeta - 1}\right] d\sigma. \tag{26.14}$$

The above formula may be put in a slightly different form if we subtract from its right-hand side the expression

$$\frac{1}{2\pi}\int\limits_{\gamma^+}\omega(\sigma)\,\bar{\omega}(\sigma)\left[\frac{1}{\sigma-\zeta}+\frac{1}{\sigma\zeta-1}\right]d\sigma+$$
$$+\frac{1}{2\pi}\int\limits_{\gamma^-}\overline{\omega(\sigma)}\,\omega(\bar{\sigma})\left[\frac{1}{\sigma-\zeta}+\frac{\zeta}{\sigma\zeta-1}-\frac{2}{\sigma}\right]d\sigma$$

which is identically zero.

After performing some elementary transformations and rejecting the constant term, we find

$$f(\zeta)=i\omega(\zeta)\,\bar{\omega}(\zeta)+$$
$$+\frac{1}{2\pi}\int\limits_{\gamma^+}\omega(\sigma)\left[\overline{\omega(\sigma)}-\bar{\omega}(\sigma)\right]\left[\frac{1}{\sigma-\zeta}+\frac{\zeta}{\sigma\zeta-1}\right]d\sigma. \quad (26.15)$$

Formula (26.15) has been obtained in a different way by E. Deutsch [19].

Examples. As an example, consider the case of a mapping effected by the function

$$\omega(\zeta)=R_0\left(\zeta+\frac{m}{\zeta}\right), \quad (26.16)$$

where m is a real number.

The cross section of the bar (in the z plane) represents the upper half of a confocal elliptical ring whose centre coincides with the origin and the major axis of symmetry is coincident with the real axis when $m>0$. The semi-axes, c_j and c_j', of the ellipses are determined by the equalities

$$c_j=R_0\left(\rho_j+\frac{m}{\rho_j}\right), \qquad c_j'=R_0\left(\rho_j-\frac{m}{\rho_j}\right) \qquad (j=1,2). \quad (26.17)$$

By a suitable choice of the radii of the circumferences and the parameter m, we can obtain, with the help of formula (26.16), a number of sections of a special kind which are of great practical interest.

Thus, for example, when $m=1$, $\rho_2=1$, the cross section of the bar represents the half of a solid ellipse which is obtained by cutting it along the major axis; when $m=1$, $\rho_2=1/\rho_1$, the section of the bar is a solid ellipse cut along two straight-line segments of the real axis which connect the foci to the vertices. If, further, we remove from a solid ellipse a segment connecting its foci and then cut the ellipse along its minor axis, the resulting figure, a semi-ellipse with a cut, corresponds to mapping (26.16) with $m=-1$, $\rho_2=1$. Finally, when $m=0$ we have a circular half-ring or a semicircle.

In the case under consideration, on the basis of formulas (26.8) to (26.10) we find, after elementary computations, the complex torsion function in the form

$$f(\zeta)=\frac{iR_0^2}{2}\left(\zeta^2+\frac{m}{\zeta^2}\right)-\frac{8R_0^2}{\pi}\sum_{n=0}^{\infty}\left[a_{2n+1}\left(\frac{\zeta}{\rho_1}\right)^{2n+1}+b_{2n+1}\left(\frac{\rho_2}{\zeta}\right)^{2n+1}\right], \tag{26.18}$$

where

$$\left.\begin{aligned} a_{2n+1}&=\frac{(\rho_1^2-m)^2-(\rho_2^2-m)^2R^{2n-1}}{(2n+3)(4n^2-1)(1-R^{4n+2})\rho_1^2},\\ b_{2n+1}&=\frac{(\rho_1^2-m)^2R^{2n+3}-(\rho_2^2-m)^2}{(2n+3)(4n^2-1)(1-R^{4n+2})\rho_2^2}\end{aligned}\right\} \tag{26.19}$$

with $R=\rho_2/\rho_1$.

By using formulas (26.12) and (26.18), we obtain the following formula for the determination of the torsional rigidity:

$$D=\mu\left(\pi\delta_1-\frac{\delta_2}{\pi}\right)-\mu\frac{16}{\pi}\sum_{n=0}^{\infty}\frac{\delta_2R^{4n+2}-\delta_3R^{2n+1}}{(2n+1)(2n-1)^2(2n+3)^2(1-R^{4n+2})}, \tag{26.20}$$

where

$$\delta_1=(\rho_1^2-\rho_2^2)\left[\frac{1}{2}(\rho_1^2+\rho_2^2)\left(1+\frac{m^4}{\rho_1^4\rho_2^4}\right)-m\left(1+\frac{m^2}{\rho_1^2\rho_2^2}\right)\right],$$

$$\delta_2=4\left[12m^2-4m\left(1+\frac{m^2}{\rho_1^2\rho_2^2}\right)(\rho_1^2+\rho_2^2)+(\rho_1^4+\rho_2^4)\left(1+\frac{m^4}{\rho_1^4\rho_2^4}\right)\right],$$

$$\delta_3=8\left[4m^2+\left(1+\frac{m^4}{\rho_1^4\rho_2^4}\right)\rho_1^2\rho_2^2-2m\left(1+\frac{m^2}{\rho_1^2\rho_2^2}\right)(\rho_1^2+\rho_2^2)+\right.$$
$$\left.+\frac{m^4}{\rho_1^2\rho_2^2}(\rho_1^4+\rho_2^4)\right].$$

By calculating also the contour values of the shearing stress components defined by formulas (26.11) and (26.18), we have on the straight-line parts of the section contour

$$T_\rho=\frac{-8\tau\mu|\xi|}{\pi(\xi^2-m)}\sum_{n=0}^{\infty}(2n+1)\left[a_{2n+1}\left(\frac{\xi}{\rho_1}\right)^{2n+1}-b_{2n+1}\left(\frac{\rho_2}{\xi}\right)^{2n+1}\right], \tag{26.21}$$

$$\rho_2\leqslant|\xi|\leqslant\rho_1 \qquad (\eta=0).$$

On the curved parts of the boundary we have

$$T_\vartheta=2\frac{\tau\mu}{\delta_j^*\rho_j}\left[(\rho_j^4-m^2)\sin^2\vartheta+\frac{2q_j}{\pi}(\rho_j^2-m)^2\times\right.$$
$$\left.\times\left(1-\cos\vartheta\ln\cot\frac{\vartheta}{2}\right)\sin\vartheta+8\frac{\rho_j^2}{\pi}\sum_{n=0}^{\infty}(2n+1)a_{2n+1}^{(j)}\sin(2n+1)\vartheta\right],$$

$$\delta_j^*=\sqrt{\rho_j^4-2m\rho_j^2\cos2\vartheta+m^2}, \quad q_1=1, \quad q_2=-1, \quad 0<\vartheta<\pi. \tag{26.22}$$

The above formulas enable one to determine explicitly singular terms in the stresses near the vertex of a re-entrant angle of the contour.

Thus, for example, in the case of a solid ellipse cut along two straight-line segments of the real axis connecting the foci to the vertices ($m = 1$, $\rho_2 = 1/\rho_1$) we have, on the basis of formulas (26.11) and (26.16), for points x lying on the real axis and close to the vertex of one of the re-entrant angles

$$\tau_{xz} = \frac{K_1}{\sqrt{s}} + O(1), \qquad \tau_{yz} = O(1),$$

where s denotes a small distance from x to the vertex of the angle

$$K_1 = -\tau\mu R_0^{3/2}\frac{8}{\pi}\sum_{n=0}^{\infty}\frac{(1-\rho_1^2)^2\rho_1^{2n-1}}{(2n-1)(2n+3)(1-\rho_1^{4n+2})} \tag{26.23}$$

and

$$\rho_1 = \frac{c_1 + c_1'}{2R_0}.$$

In the case when the cross section of the bar represents a semi-ellipse with a cut, i.e., when $m = -1$, $\rho_2 = 1$, we have near the re-entrant angle

$$\tau_{xz} = O(1), \qquad \tau_{yz} = \frac{K_2}{\sqrt{s}} + O(1),$$

where

$$K_2 = \tau\mu R_0^{3/2}\frac{8}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{(1+\rho_1^2)^2\rho_1^{2n-1} - 4(\rho_1^{4n+2}+1)}{(2n-1)(2n+3)(\rho_1^{4n+2}-1)}. \tag{26.24}$$

Consequently, in both these cases the absolute value of the shearing stress vector

$$T = \sqrt{\tau_{xz}^2 + \tau_{yz}^2}$$

near the re-entrant angles of the contour is represented as

$$T = \frac{K}{\sqrt{s}} + O(1), \tag{26.25}$$

where the constant K, the so-called stress intensity factor, is defined by formulas (26.23) and (26.24).

The specific examples considered above, as well as some others similar to them, have been solved by different methods by N. Kh. Arutyunyan and B. L. Abramyan [3], V. I. Blokh [9], V. V. Novozhilov [97], D. I. Sherman [127] and A. Herzig [34]. In particular, the foregoing solution in the case of a circular half-ring is identical with that given in the work of V. V. Novozhilov and D. I. Sherman. The solution for a semi-circle ($m = 0$, $\rho_2 = 0$) is contained in the above-mentioned work of A. Herzig (see also A. E. H. Love [74]).

The nature of the shearing stress distribution near the vertices of re-entrant angles has been investigated in a different way by G. C. Sih [129].

2°. **Torsion of composite rods.** Suppose that a twisted rod is made up of two homogeneous and isotropic bars with different elastic properties bonded together along a part of their lateral surfaces.

As usual, it is assumed that the stress vector and the displacement components remain continuous across the interface.

Let μ_1 and μ_2 be the shearing moduli of the component materials, S_1 and S_2 the cross sections of the bars, S the total section of the composite body ($S = S_1 + S_2$), L the trace of the interface on the z plane of the section. We require that the function

$$z = \omega(\zeta)$$

which effects a conformal mapping of the region S onto the unit circle in the plane of the variable $\zeta = \xi + i\eta$ should transform the line of separation L into the diameter of the circle $-1 \leqslant \xi \leqslant 1$, $\eta = 0$ which is denoted, as before, by l. We also retain the previous notation for the semi-circles in the ζ plane and their boundaries.

The problem under consideration is then reduced to finding two functions, $F_1(\zeta)$ and $F_2(\zeta)$, which are holomorphic in the regions Σ^+ and Σ^-, respectively, from the boundary conditions

$$F_1(\sigma) + \overline{F_1(\sigma)} = \omega(\sigma)\,\overline{\omega(\sigma)} \quad \text{on} \quad \gamma^+, \tag{26.26}$$

$$F_2(\sigma) + \overline{F_2(\sigma)} = \omega(\sigma)\,\overline{\omega(\sigma)} \quad \text{on} \quad \gamma^-, \tag{26.27}$$

$$F_1(\sigma) - \overline{F_1(\sigma)} = F_2(\sigma) - \overline{F_2(\sigma)} \quad \text{on} \quad l, \tag{26.28}$$

$$\mu_1[F_1(\sigma) + \overline{F_1(\sigma)}] - \mu_2[F_2(\sigma) + \overline{F_2(\sigma)}] = (\mu_1 - \mu_2)\,\omega(\sigma)\,\overline{\omega(\sigma)} \quad \text{on} \quad l. \tag{26.29}$$

After finding $F_j(\zeta)$, the corresponding torsion functions are determined by the formula

$$f_j(\zeta) = iF_j(\zeta) \qquad (j = 1, 2). \tag{26.30}$$

By adding equalities (26.28) and (26.29) together, we have

$$(\mu_1 + \mu_2)\,F_1(\sigma) + (\mu_1 - \mu_2)\,\overline{F_1(\sigma)} - 2\mu_2 F_2(\sigma) =$$
$$= (\mu_1 - \mu_2)\,\omega(\sigma)\,\overline{\omega(\sigma)} \quad \text{on} \quad l. \tag{26.31}$$

We extend the definition of the function $F_1(\zeta)$ to the region Σ^- by the equality

$$(\mu_1+\mu_2)\,F_1(\zeta)=2\mu_2F_2(\zeta)-(\mu_1-\mu_2)\,\overline{F_1}(\zeta)+ \\ +(\mu_1-\mu_2)\,\omega(\zeta)\,\overline{\omega}(\zeta)\qquad(\zeta\in\Sigma^-).\qquad(26.32)$$

Then, on the basis of formula (26.31) it is easily verified that the function $F_1(\zeta)$ defined by formula (26.32) analytically continues the values of $F_1(\zeta)$ from the upper semi-circle Σ^+ into the lower semi-circle through the segment l. This extended function holomorphic in the entire circle is denoted by $F(\zeta)$.

Equality (26.32) enables us to express the unknown function $F_2(\zeta)$ in Σ^- through $F(\zeta)$ as follows:

$$2\mu_2F_2(\zeta)=(\mu_1+\mu_2)\,F(\zeta)+(\mu_1-\mu_2)\,\overline{F}(\zeta)- \\ -(\mu_1-\mu_2)\,\omega(\zeta)\,\overline{\omega}(\zeta).\qquad(26.33)$$

The boundary conditions (26.26) and (26.27) may now be put in the form

$$F(\sigma)+\overline{F(\sigma)}=\omega(\sigma)\,\overline{\omega(\sigma)}\quad\text{on}\quad\gamma^+,\qquad(26.34)$$

$$(\mu_1+\mu_2)\,[F(\sigma)+\overline{F(\sigma)}]+(\mu_1-\mu_2)\,[\overline{F}(\sigma)+F(\overline{\sigma})]= \\ =2\mu_2\omega(\sigma)\,\overline{\omega(\sigma)}+(\mu_1-\mu_2)\,[\omega(\sigma)\,\overline{\omega}(\sigma)+\omega(\overline{\sigma})\,\overline{\omega(\sigma)}]\text{ on }\gamma^-.\qquad(26.35)$$

We replace σ by $\overline{\sigma}$ in formula (26.34), assuming that σ belongs to γ^-. Then

$$F(\overline{\sigma})+\overline{F}(\sigma)=\omega(\overline{\sigma})\,\overline{\omega}(\sigma)\text{ on }\gamma^-.$$

The above equality permits simplification of conditions (26.34), (26.35). For the function $F(\zeta)$ we finally have the following boundary condition:

$$F(\sigma)+\overline{F(\sigma)}=\begin{cases}\omega(\sigma)\,\overline{\omega(\sigma)}\quad\text{on}\quad\gamma^+,\\ \dfrac{2\mu_2}{\mu_1+\mu_2}\,\omega(\sigma)\,\overline{\omega(\sigma)}+\dfrac{\mu_1-\mu_2}{\mu_1+\mu_2}\,[\omega(\sigma)\,\overline{\omega}(\sigma)+\\ \qquad+\omega(\overline{\sigma})\,\overline{\omega(\sigma)}-\omega(\overline{\sigma})\,\overline{\omega}(\sigma)]\quad\text{on}\quad\gamma^-.\end{cases}\qquad(26.36)$$

We have thus obtained the Dirichlet problem for a circle. We shall solve it by using the well-known Schwarz integral

$$F(\zeta)=\frac{1}{2\pi i}\int\limits_{\gamma}[F(\sigma)+\overline{F(\sigma)}]\,\frac{d\sigma}{\sigma-\zeta}+\text{constant}\qquad(\gamma=\gamma^++\gamma^-).$$

After some transformations, rejecting the constant term which has no effect on the state of stress, we have, for the solution of (26.36),

$$F(\zeta)=\frac{\mu_2}{\pi i\,(\mu_1+\mu_2)}\int\limits_{\gamma}\omega(\sigma)\,\overline{\omega(\sigma)}\,\frac{d\sigma}{\sigma-\zeta}+ \\ +\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\,\frac{1}{2\pi i}\int\limits_{\gamma'}\overline{\omega(\sigma)}\,[\omega(\sigma)-\omega(\overline{\sigma})]\left[\frac{1}{\sigma-\zeta}+\frac{\zeta}{\sigma\zeta-1}\right]d\sigma.\qquad(26.37)$$

It can easily be verified that the functions $F_1(\zeta)$, $F_2(\zeta)$ determined from formulas (26.33) and (26.37) satisfy all conditions specified above. The problem is solved.

Examples. 1. Let the region of the section of a rod be made up of two dissimilar semi-circles bonded together along their common diameter. In this case $z = \zeta$ and, on the basis of formulas (26.30), (26.33) and (26.37), the torsion function is found to be

$$f_j(\zeta) = \frac{1}{2\pi}\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\left[(3-2j)\pi i\zeta^2 + {} \right.$$
$$\left. + 2\left(\zeta+\frac{1}{\zeta}\right)+\left(\zeta-\frac{1}{\zeta}\right)^2 \ln\frac{1-\zeta}{1+\zeta}\right] \quad (j=1,\ 2). \qquad (26.38)$$

Formula (26.38) has been indicated, and found in an entirely different way, by D. I. Sherman [127] and by L. Dumitrescu and C. Stănescu [20].

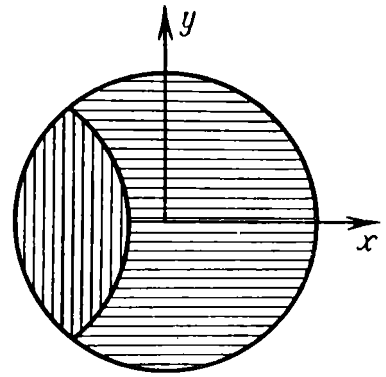

Fig. 2

2. Suppose, now, that the region of the cross section of a composite body represents a circle of unit radius and the interface of the media is a circular arc of the same radius which intersects the boundary of the complete region at right angles (Fig. 2). The mapping function here is

$$z = a\left(\frac{2}{a+i\zeta}-1\right), \qquad a = 1+\sqrt{2},$$

and the corresponding torsion functions are of the form

$$f_j(\zeta) = 2\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\left[(3-2j)\frac{i\sqrt{2}\zeta^2}{a^2+\zeta^2} + \frac{\zeta(1+\zeta)^2}{\zeta^4+6\zeta^2+1} + {} \right.$$
$$\left. + \frac{1}{\pi}\frac{(1-\zeta^2)^2}{\zeta^4+6\zeta^2+1}\ln\frac{1-\zeta}{1+\zeta}\right] \quad (j=1,\ 2). \qquad (26.39)$$

If $\mu_1 = 0$ is substituted in the above formula, we obtain a solution for a lens shaped section, and when $\mu_2 = 0$, we have a solution for a circular lune. These solutions have been previously indicated in the work of Ya. I. Burak [16], Ya. I. Burak and M. Ya. Leonov [17] and Ya. S. Uflyand [135].

When $\mu_1 = \mu_2$, we obtain the well-known solution of the torsion problem for a circular cylindrical bar.

3°. **Continuation.** In conclusion we consider a less elementary example. Let a sectionally homogeneous rod be made up of two dissimilar (in elastic properties) bars of the same cross section in the shape of a quarter unit circle bonded together along their plane faces. The complete section of such a rod will obviously fill the region of a semi-circle Σ^+.

The co-ordinate system is taken as shown in Fig. 3. Let Σ_1^+ and Σ_2^+ be the regions of the bar sections, γ_1^+ and γ_2^+ the curved parts and l_1, l_2, l_0 the straight-line parts of their boundaries, with l_0 denoting the trace of the interface of the different media (see Fig. 3).

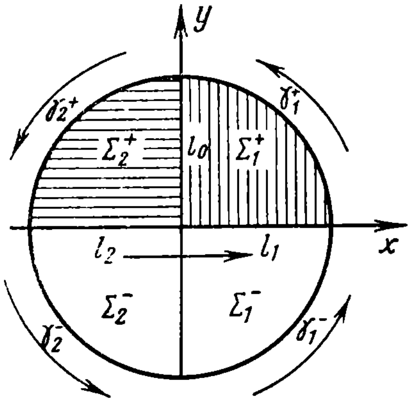

Fig. 3

The boundary conditions for determining the functions $F_1(z)$ and $F_2(z)$ holomorphic, respectively, in the regions Σ_1^+ and Σ_2^+ are of the form

$$F_1(t)+\overline{F_1(t)}=t\bar{t} \quad \text{on} \quad \gamma_1^+ + l_1, \tag{26.40}$$

$$F_2(t)+\overline{F_2(t)}=t\bar{t} \quad \text{on} \quad \gamma_2^+ + l_2, \tag{26.41}$$

$$F_1(t)-\overline{F_1(t)}=F_2(t)-\overline{F_2(t)} \quad \text{on} \quad l_0, \tag{26.42}$$

$$\mu_1[F_1(t)+\overline{F_1(t)}]-\mu_2[F_2(t)+ \\ +\overline{F_2(t)}]=(\mu_1-\mu_2)t\bar{t} \text{ on } l_0. \tag{26.43}$$

Proceeding to the solution, add equalities (26.42) and (26.43). We have

$$(\mu_1+\mu_2)F_1(t)+(\mu_1-\mu_2)\overline{F_1(t)}-2\mu_2F_2(t)= \\ =(\mu_1-\mu_2)t\bar{t} \text{ on } l_0. \tag{26.44}$$

Define the function $F_1(z)$ in the region Σ_2^+ by the equality

$$(\mu_1+\mu_2)F_1(z)=2\mu_2F_2(z)-(\mu_1-\mu_2)\overline{F}_1(-z)-(\mu_1-\mu_2)z^2 \\ (z\in\Sigma_2^+). \tag{26.45}$$

Condition (26.44) then allows us to state that the function $F_1(z)$, just defined, analytically continues the values of the unknown function $F_1(z)$ holomorphic in Σ_1^+ into the region Σ_2^+, symmetrical about the y axis, through the segment l_0. The function $F_1(z)$ is thus holomorphic in the entire upper semi-circle Σ^+.

Assuming

$$F_1(z)=\frac{1}{2}z^2+F(z), \tag{26.46}$$

we find from formula (26.45)

$$2\mu_2F_2(z)=(\mu_1+\mu_2)F(z)+(\mu_1-\mu_2)\overline{F}(-z)+ \\ +(2\mu_1-\mu_2)z^2 \quad (z\in\Sigma_2^-). \tag{26.47}$$

Taking into account the above two equalities, we represent conditions (26.40) and (26.41) as follows:

$$F(t)+\overline{F(t)}=t\bar{t}-\frac{1}{2}(t^2+\bar{t}^2) \quad \text{on} \quad \gamma_1^+ + l_1, \tag{26.48}$$

$$(\mu_1+\mu_2)[F(t)+\overline{F(t)}]+(\mu_1-\mu_2)[\overline{F}(-t)+F(-\bar{t})]= \\ =2\mu_2t\bar{t}-(2\mu_1-\mu_2)(t^2+\bar{t}^2) \quad \text{on} \quad \gamma_2^+ + l_2. \tag{26.49}$$

Replace t by $(-\bar{t})$ in formula (26.48), assuming the point t to lie on the curve $\gamma_2^+ + l_2$. We have

$$F(-\bar{t})+\bar{F}(-t)=t\bar{t}-\frac{1}{2}(t^2+\bar{t}^2) \quad \text{on} \quad \gamma_2^+ + l_2.$$

On the basis of this equality formula (26.49) becomes

$$(\mu_1+\mu_2)\,[F(t)+\overline{F(t)}]=$$
$$=-(\mu_1-3\mu_2)\,t\bar{t}-\frac{1}{2}(3\mu_1-\mu_2)(t^2+\bar{t}^2) \quad \text{on} \quad \gamma_2^+ + l_2. \qquad (26.50)$$

We now define a new function $\Phi(z)$, holomorphic in Σ^+, by the equality

$$\Phi(z)=F(z)+\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\,\frac{2z^2}{\pi i}\ln z \qquad (0\leqslant \operatorname{Im}\ln z\leqslant\pi). \qquad (26.51)$$

According to formulas (26.48) and (26.50) we have the following boundary conditions for the function $\Phi(z)$:

$$\Phi(t)+\overline{\Phi(t)}=1-\left[\frac{1}{2}-\frac{2\vartheta}{\pi}\,\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\right](t^2+\bar{t}^2) \quad \text{on} \quad \gamma_1^+, \qquad (26.52)$$

$$\Phi(t)+\overline{\Phi(t)}=-\frac{\mu_1-3\mu_2}{\mu_1+\mu_2}-\left[\frac{1}{2}\,\frac{3\mu_1-\mu_2}{\mu_1+\mu_2}-\frac{2\vartheta}{\pi}\,\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\right](t^2+\bar{t}^2) \quad \text{on} \quad \gamma_2^+, \qquad (26.53)$$

$$\Phi(t)+\overline{\Phi(t)}=0 \quad \text{on } l, \quad l=l_1+l_2. \qquad (26.54)$$

By formula (26.54), the function $\Phi(z)$ can be continued analytically into the lower semi-circle Σ^- through l, and hence it is a holomorphic function in the entire circle $|z|<1$. Its values in the lower semi-circle Σ^- are given by the formula

$$\Phi(z)=-\bar{\Phi}(z) \quad \text{when} \quad z \text{ is in } \Sigma^-, \qquad (26.55)$$

whence

$$\overline{\Phi(z)}=-\Phi(\bar{z}) \quad \text{when } z \text{ is in } \Sigma^+.$$

If $\overline{\Phi(t)}$ in equalities (26.52), (26.53) is now replaced by the above value, these equalities are written as

$$\Phi(t)-\Phi(\bar{t})=\begin{cases}\Omega_1(t) & \text{on} \quad \gamma_1^+,\\ \Omega_2(t) & \text{on} \quad \gamma_2^+,\end{cases} \qquad (26.56)$$

where $\Omega_1(t)$ and $\Omega_2(t)$ stand for the right-hand sides of relations (26.52) and (26.53), respectively.

By replacing the argument t by $\bar{t}$ in formula (26.56), we obtain the corresponding conditions on the symmetrical circular arcs forming the lower semi-circumference:

$$\Phi(t)-\Phi(\bar{t})=\begin{cases}-\Omega_1(\bar{t}) & \text{on} \quad \gamma_1^-,\\ -\Omega_2(\bar{t}) & \text{on} \quad \gamma_2^-.\end{cases} \qquad (26.57)$$

Thus, after the double application of the analytic continuation of the unknown functions we arrive at the problem of determining the single function $\Phi(z)$, holomorphic in the unit circle, from conditions (26.56) and (26.57).

The use of the power series method determines $\Phi(z)$ as

$$\Phi(z)=\sum_{n=0}^{\infty}\left[a_{2n+1}z^{2n+1}+a_{2(2n+1)}z^{2(2n+1)}\right],$$

where

$$a_{2k+1}=\frac{1}{\pi i}\,\frac{2\mu_2}{\mu_1+\mu_2}\left[\frac{2}{2k+1}-\frac{1}{2k-1}-\frac{1}{2k+3}\right]\qquad (k=0,\,1,\,2,\,\ldots),$$

$$a_2=\frac{3}{2\pi i}\,\frac{\mu_1-\mu_2}{\mu_1+\mu_2},$$

$$a_{2(2k+1)}=\frac{1}{\pi i}\,\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\left[\frac{2}{2k+1}-\frac{1}{2k}-\frac{1}{2(k+1)}\right]\qquad (k=1,\,2,\,\ldots).$$

To calculate the torsion function $f_j(z)$ from $\Phi(z)$, we turn to equalities (26.30), (26.46), (26.47) and (26.51). We have, finally,

$$f_j(z)=-\frac{i}{2}\,\frac{3\mu_1-5\mu_2}{\mu_1+\mu_2}\,z^2+\frac{2}{\pi}\left(1-\frac{\mu_j}{\mu_1+\mu_2}\right)\left[z+\frac{1}{z}+\frac{1}{2}\left(z-\frac{1}{z}\right)^2\times\right.$$
$$\left.\times\ln\frac{1-z}{1+z}\right]+\frac{1}{\pi}\,\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\left[2z^2\,(i\pi j-\ln z)+\frac{1}{2}\left(z^2+\frac{1}{z^2}\right)\times\right.$$
$$\left.\times\ln(1-z^4)-\ln\frac{1-z^2}{1+z^2}\right]\qquad (j=1,\,2).\qquad (26.58)$$

Taking into account that

$$L_1=\gamma_1^{+}+l_1+l_0,\qquad L_2=\gamma_2^{+}+l_2+l_0,$$

the torsional rigidity is obtained, after elementary computations, as

$$D=\frac{\mu_1\mu_2}{\mu_1+\mu_2}\left(\pi-\frac{8}{\pi}\right)+\frac{(\mu_1-\mu_2)^2}{\mu_1+\mu_2}\left(\frac{\pi}{6}+\frac{2}{\pi}\ln 2\right).\qquad (26.59)$$

The shearing stress components at any point of the sections Σ_j^{+} are determined by the formula

$$T_r-iT_\vartheta=\tau\mu_j e^{i\vartheta}\,[f_j'(z)-i\overline{z}]\qquad (j=1,\,2),\qquad (26.60)$$

where $f_j(z)$ is given by (26.58).

Based on formula (26.60), calculate the stresses on the contours of the sections. We have:

on the curved part of the boundary

$$T_\vartheta=\tau\mu_j\left\{1-\left[2-j+(1-j)\frac{3\mu_2-5\mu_1}{\mu_1+\mu_2}\right]\cos 2\vartheta+\right.$$
$$+\frac{2}{\pi}\,\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\,[2\vartheta\cos 2\vartheta-\sin 2\vartheta\ln(2\sin 2\vartheta)]-$$
$$\left.-\frac{8}{\pi}\left(1-\frac{\mu_j}{\mu_1+\mu_2}\right)\left[\sin\vartheta-\frac{1}{2}\sin 2\vartheta\ln\cot\frac{\vartheta}{2}\right]\right\},\qquad 0<\vartheta<\pi.$$
$$(26.61)$$

From this, in particular, we find

$$\left.\begin{aligned} T_\vartheta &= \frac{4\tau\mu_1\mu_2}{\mu_1+\mu_2}\left(1-\frac{2}{\pi}\right) \quad \text{when} \quad \vartheta=\frac{\pi}{2}\pm 0, \\ T_\vartheta &= 0 \quad \text{when} \quad \vartheta = 0,\ \pi. \end{aligned}\right\} \tag{26.62}$$

On the straight-line part of the boundary

$$T_r = \tau\mu_j\left\{\frac{2}{\pi}\left(1-\frac{\mu_j}{\mu_1+\mu_2}\right)\left[2\left(1-\frac{1}{x^2}\right)+\left(x-\frac{1}{x^3}\right)\ln\frac{1-x}{1+x}\right]-\right.$$
$$\left.-\frac{1}{\pi}\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\left[2|x|\ln x^2-\left(|x|-\frac{1}{|x|^3}\right)\ln(1-x^4)\right]\right\} \quad (0<|x|<1). \tag{26.63}$$

In particular,

$$T_r = \frac{16\tau}{3\pi}\frac{\mu_1\mu_2}{\mu_1+\mu_2} \text{ when } r=\pm 0, \qquad T_r = 0 \text{ when } r=\pm 1. \tag{26.64}$$

It is noted, finally, that when $\mu_1 = \mu_2$ and $\mu_2 = 0$ the solutions obtained above are identical, respectively, with the well-known solutions for a homogeneous rod in the shape of a semi-circle and a quarter circle found in the above-mentioned work of A. Herzig [34].

27. Continuation

The method indicated in Sec. 22 can be applied in a slightly modified form to the solution of problems in cases where our basic requirement on analytic continuation of the mapping function $\omega(\zeta)$ is not fulfilled. This is exemplified by the following well-known solution of the torsion problem for a sector of a circular ring.

Let the sector under consideration with angular span $\alpha\pi$ ($0 < \alpha \leqslant 2$) occupy in the z plane the region $r_2 \leqslant r \leqslant r_1$, $0 \leqslant \vartheta \leqslant \alpha\pi$ ($z = re^{i\vartheta}$), where r_1 and r_2 are the radii of circles bounding the ring and centred at the origin.

Make a transformation

$$z = \omega(\zeta) = \zeta^\alpha \qquad (0 < \alpha \leqslant 2) \tag{27.1}$$

which maps the sector of the ring onto a circular half-ring Σ^+.

In the ζ plane we introduce a new function holomorphic in Σ^+:

$$\Phi(\zeta) = F(\zeta) - \frac{1}{2}[1 - ig(\alpha)]\,\zeta^{2\alpha}, \tag{27.2}$$

where

$$g(\alpha) = \tan\alpha\pi, \qquad 0<\alpha\leqslant 2, \qquad \alpha \neq \frac{1}{2},\ \frac{3}{2}. \tag{27.3}$$

Conditions (26.2) then become

$$\left.\begin{aligned} \Phi(\sigma)+\overline{\Phi(\sigma)} &= \rho_j^{2\alpha}-\frac{1}{2}\left(\sigma^{2\alpha}+\bar{\sigma}^{2\alpha}\right)+ \\ &\qquad +ig(\alpha)\frac{1}{2}\left(\sigma^{2\alpha}-\bar{\sigma}^{2\alpha}\right) \text{ on } \gamma^{+}, \\ \Phi(\sigma)+\overline{\Phi(\sigma)} &= 0 \quad \text{on } l \qquad (j=1,\,2). \end{aligned}\right\} \tag{27.4}$$

It will be recalled that γ^{+} and l denote, respectively, the sets of curved and straight-line parts of the boundary of the upper half-ring.

On the basis of the second of conditions (27.4) we conclude, by perfect analogy with the above, that the function $\Phi(\zeta)$ is holomorphic in the entire ring $\rho_2 < |\zeta| < \rho_1$ and its value in the lower half-ring Σ^{-} is given by formula (26.55).

Conditions (27.4) for this function $\Phi(\zeta)$ take the form

$$\Phi(\sigma)-\Phi(\bar{\sigma})=\rho_j^{2\alpha}-\frac{1}{2}(1-ig)\,\sigma^{2\alpha}-\frac{1}{2}(1+ig)\,\bar{\sigma}^{2\alpha} \text{ on } \gamma^{+}. \tag{27.5}$$

By extending condition (27.5) to the lower part of the boundary of the ring, we obtain an equality

$$\begin{aligned} \Phi(\sigma)-\Phi(\bar{\sigma}) = &-\rho_j^{2\alpha}+\frac{1}{2}(1+ig)\,\sigma^{2\alpha}e^{-i4\alpha\vartheta}+ \\ &+\frac{1}{2}(1-ig)\,\bar{\sigma}^{2\alpha}e^{i4\alpha\vartheta} \text{ on } \gamma^{-} \end{aligned} \tag{27.6}$$

which, together with conditions (27.5), determines the boundary condition for $\Phi(z)$ as follows:

$$\Phi(\sigma)-\Phi(\bar{\sigma})=\begin{cases} \Omega_j'(\sigma) & \text{on } \gamma^{+}, \\ \Omega_j''(\sigma) & \text{on } \gamma^{-}, \end{cases} \tag{27.7}$$

where $\Omega_j'(\sigma)$ and $\Omega_j''(\sigma)$ stand for the right-hand sides of equalities (27.5) and (27.6), respectively.

Solving this problem by the power series method, we find an expression for the unknown function:

$$\Phi(\zeta)=\sum_{n=0}^{\infty}\left[a_{2n+1}\zeta^{2n+1}+b_{2n+1}\zeta^{-(2n+1)}\right], \tag{27.8}$$

where

$$a_{2n+1}=\frac{A_{2n+1}^{(1)}-A_{2n+1}^{(2)}R^{2n+1}}{\left[1-R^{4n+2}\right]\rho_1^{2n+1}},$$

$$b_{2n+1}=\rho_2^{2n+1}\,\frac{A_{2n+1}^{(1)}R^{2n+1}-A_{2n+1}^{(2)}}{1-R^{4n+2}}, \qquad R=\frac{\rho_2}{\rho_1},$$

$$A_{2n+1}^{(j)}=\frac{\rho_j^{2\alpha}}{\pi i}\left[\frac{2}{2n+1}-\frac{1}{2n-2\alpha+1}-\frac{1}{2n+2\alpha+1}\right]. \tag{27.9}$$

The corresponding torsion function is represented as

$$f(\zeta)=\frac{1}{2}(i+g)\,\zeta^{2\alpha}-\frac{8}{\pi}\sum_{n=1}^{\infty}{}^{*}\,\frac{\alpha^2\rho_1^{2\alpha}}{n(n^2-4\alpha^2)}\times$$

$$\times\left[\frac{1-R^{n+2\alpha}}{1-R^{2n}}\left(\frac{\zeta}{\rho_1}\right)^n-\frac{R^{2\alpha}-R^n}{1-R^{2n}}\left(\frac{\rho_2}{\zeta}\right)^n\right], \qquad (27.10)$$

where an asterisk on Σ indicates that the summation index takes odd values ($n = 1, 3, 5, \ldots$).

In the above discussion we excluded the cases of $\alpha = 1/2$ and $\alpha = 3/2$. On passing in formula (27.10) to the limit as $\alpha \to 1/2$ or $\alpha \to 3/2$, we have

$$f(\zeta)=\frac{\zeta^{2\alpha}}{\pi}\left[\alpha+\frac{\pi i}{2}-\ln\frac{\zeta}{\rho_1}\right]-\left(\zeta^{2\alpha}+\frac{\rho_1^{4\alpha}}{\zeta^{2\alpha}}\right)\frac{R^{4\alpha}\ln R}{\pi(1-R^{4\alpha})}+$$

$$+\left(\alpha-\frac{1}{2}\right)\frac{9\rho_1^2}{4\pi}\left[\zeta+R^2\left(\zeta+\frac{\rho_1^2}{\zeta}\right)\right]-$$

$$-\frac{8}{\pi}\sum_{n=2(\alpha+1)}^{\infty}{}^{*}\,\frac{\alpha^2\rho_1^{2\alpha}}{n(n^2-4\alpha^2)}\left[\frac{1-R^{n+2\alpha}}{1-R^{2n}}\left(\frac{\zeta}{\rho_1}\right)^n-\right.$$

$$\left.-\frac{R^{2\alpha}-R^n}{1-R^{2n}}\left(\frac{\rho_2}{\zeta}\right)^n\right] \qquad \left(\alpha=\frac{1}{2},\ \frac{3}{2}\right), \qquad (27.11)$$

It is easily verified that the function $f(\zeta)$ defined by the above equality is the complex torsion function for the cases $\alpha = 1/2, 3/2$.

In passing, we note that these exceptional cases can also be considered directly. Instead of function (27.2), it is sufficient to introduce into consideration another function, also holomorphic in Σ^+, of the form

$$\Phi_1(\zeta)=F(\zeta)-\frac{1}{2}\zeta^{2\alpha}+\frac{1}{\pi i}\zeta^{2\alpha}\ln\zeta,$$

$$0\leqslant \operatorname{Im}\ln\zeta\leqslant\pi \qquad \left(\alpha=\frac{1}{2},\ \frac{3}{2}\right)$$

and for the rest to proceed as before.

For the determination of the torsional rigidity [see formula (26.12)] we find after elementary computations

$$D=\frac{8\mu\rho_1^{4\alpha}}{\pi}\sum_{n=0}^{\infty}\left\{\frac{\alpha^3(1-R^{4\alpha})}{(2n+1)^2\,(2n+2\alpha+1)^2}-\right.$$

$$\left.-\frac{8\alpha^4\,(R^{2\alpha}-R^{2n+1})^2}{(2n+1)\,(1-R^{4n+2})\,[(2n+1)^2-4\alpha^2]^2}\right\}\quad\left(\alpha\neq\frac{1}{2},\ \frac{3}{2}\right),\qquad(27.12)$$

$$D=\frac{\mu\rho_1^{4\alpha}}{\pi}\left\{(1-R^{4\alpha})\left[\frac{\pi^2}{3}\alpha-\frac{7}{4}\left(\alpha-\frac{1}{2}\right)-2\ln 2\right]-\right.$$

$$-\frac{2\alpha R^{4\alpha}}{1-R^{4\alpha}}(\ln R)^2-\left(\alpha-\frac{1}{2}\right)\alpha^4\,(1-R^2)\,R^2-$$

$$\left.-\sum_{n=2(1+\alpha)}^{\infty}{}^{*}\ \frac{64\alpha^4\,(R^{2\alpha}-R^n)^2}{n\,(n^2-4\alpha^2)^2\,(1-R^{2n})}\right\}\quad\left(\alpha=\frac{1}{2},\ \frac{3}{2}\right).\qquad(27.13)$$

By using the above formula for the complex torsion function $f(\zeta)$, we can write without much difficulty explicit expressions for contour values of the stress components. Thus, on the curved part of the boundary

$$T_\vartheta=\tau\mu\rho_j^\alpha\left\{1-\cos 2\alpha\vartheta-g\sin 2\alpha\vartheta+\frac{8\alpha}{\pi}\sum_{n=1}^{\infty}{}^{*}\ \frac{\sin n\vartheta}{n^2-4\alpha^2}-\right.$$

$$\left.-\frac{16}{\pi}\alpha\sum_{n=1}^{\infty}{}^{*}\left(\delta_{j1}R^n+\frac{\delta_{j2}}{R^{2\alpha}}\right)\frac{(R^{2\alpha}-R^n)\sin n\vartheta}{(1-R^{2n})\,(n^2-4\alpha^2)}\right\}$$

$$\left(\alpha\neq\frac{1}{2},\ \frac{3}{2}\right),\qquad(27.14)$$

$$T_\vartheta=\mu\rho_j^\alpha\frac{\tau}{\pi}\left\{\pi-(\pi-2\vartheta)\cos 2\alpha\vartheta-\sin 2\alpha\vartheta\ln(4\sin^2\vartheta)-\right.$$

$$-(2\alpha-1)\left(1-\frac{3\rho_1^2\rho_2^2}{2\rho_j^4}\right)\sin\vartheta+\frac{4\rho_2^{4\alpha}\ln R}{\rho_j^{4\alpha}(1-R^{4\alpha})}\sin 2\alpha\vartheta-$$

$$\left.-16\alpha\sum_{n=2(1+\alpha)}^{\infty}{}^{*}\left(\delta_{j1}R^n+\frac{\delta_{j2}}{R^{2\alpha}}\right)\frac{(R^{2\alpha}-R^n)\sin n\vartheta}{(1-R^{2n})\,(n^2-4\alpha^2)}\right\}$$

$$\left(\alpha=\frac{1}{2},\ \frac{3}{2}\right),\qquad(27.15)$$

where δ_{jk} is the Kronecker symbol.

On the straight-line parts of the contour

$$T_\rho = \tau\mu \left\{ g|\xi|^\alpha (\operatorname{sgn}\xi) - \frac{8}{\pi} \sum_{n=1}^{\infty}{}^{*} \frac{\alpha\rho_1^{2\alpha}}{(n^2-4\alpha^2)|\xi|^\alpha} \left[\frac{1-R^{n+2\alpha}}{1-R^{2n}} \left(\frac{\xi}{\rho_1}\right)^n + \right.\right.$$

$$\left.\left. + \frac{R^{2\alpha}-R^n}{1-R^{2n}} \left(\frac{\rho_2}{\xi}\right)^n \right] \right\} \quad \left(\rho_2 \leqslant |\xi| \leqslant \rho_1,\ \alpha \neq \frac{1}{2}, \frac{3}{2}\right), \qquad (27.16)$$

$$T_\rho = \frac{\tau\mu}{2\alpha\pi|\xi|^\alpha} \left\{ 3\alpha\left(\alpha - \frac{1}{2}\right)\xi\rho_1^2\left(1 + R^2 - \frac{\rho_2^2}{\xi^2}\right) + \right.$$

$$+ \left(1 - 2\alpha \ln\frac{\xi^2}{\rho_1^2}\right)\xi^{2\alpha} + \frac{4\alpha}{\xi^{2\alpha}}\left(\rho_1^{4\alpha} - \xi^{4\alpha}\right)\frac{R^{4\alpha}\ln R}{1-R^{4\alpha}} -$$

$$\left. - \sum_{n=2(1+\alpha)}^{\infty}{}^{*} \frac{16\alpha^2\rho_1^{2\alpha}}{n^2-4\alpha^2} \left[\frac{1-R^{n+2\alpha}}{1-R^{2n}} \left(\frac{\xi}{\rho_1}\right)^n + \frac{R^{2\alpha}-R^n}{1-R^{2n}} \left(\frac{\rho_2}{\xi}\right)^n \right] \right\}$$

$$\left(\rho_2 \leqslant |\xi| \leqslant \rho_1, \qquad \alpha = \frac{1}{2}, \frac{3}{2}\right). \qquad (27.17)$$

The problem is completely solved.

In the particular case of $R = 0$, i.e., when the region of the section of a body represents a sector of a solid circle, the above formulas may be written, for all small values of ρ, as

$$T_\rho = \frac{8\alpha\tau\mu\rho_1^{2\alpha-1}}{\pi(4\alpha^2-1)} \frac{\cos\vartheta}{\rho^{\alpha-1}} + O(\rho),$$

$$T_\vartheta = \frac{8\alpha\tau\mu\rho_1^{2\alpha-1}}{\pi(1-4\alpha^2)} \frac{\sin\vartheta}{\rho^{\alpha-1}} + O(\rho).$$

Consequently, the absolute value of the shearing stress vector in the neighbourhood of the vertex of the sector is

$$T = \frac{K}{\sqrt[\alpha]{r^{\alpha-1}}} + O\left(r^{\frac{1}{\alpha}}\right) \qquad (1 < \alpha \leqslant 2),$$

where

$$K = \frac{8\alpha\tau\mu}{\pi(4\alpha^2-1)} \sqrt[\alpha]{r_1^{2\alpha-1}}.$$

In general, it is well to note that in the foregoing particular case the above formulas are considerably simplified and the solution of the problem can be obtained for a series of values of the angular span $\left(\text{for example, } \alpha = \frac{1}{2}, 1, \frac{3}{2}, 2\right)$ in terms of elementary functions.

The solutions of these specific problems, and also of similar ones, derived by different methods may be found in the work, to which reference has been made in Par. 1° of the preceding section, of N. Kh. Arutyunyan and B. L. Abramyan [3], D. I. Sherman [127], V. V. Novozhilov [97] and A. Herzig [34] (see also P. Kandler [55]). The solution of the torsion problem for a sector of a ring considered by us is obtained in a different way in the aforementioned monograph by the first two authors. In the same book there is a rather comprehensive bibliography of the relevant problems.

28. An Alternative Method of Solving Plane Problems for a Semi-circle (Influence Function Method)

Biharmonic problems for a semi-circle can be solved by an alternative, and more elegant (but apparently not more efficient), method which leads to a singular integral equation of the special kind. It will be developed here by reference to the mixed problem of the bending of a plate when it is fixed over a part of the boundary and free over the remainder.*

Thus, we consider a thin isotropic plate in the shape of a semicircle bent by a uniform normal load q distributed over the middle surface. We place the elastic medium in the plane of the variable $z = x + iy$ so that it occupies the upper half of a unit circle centred at the origin. The curved part of the contour, which is given to be fixed, is denoted by γ and the diameter of the circle, which is assumed to be free, by l.

By the argument used in Sec. 1 (p. 14), the problem under consideration is reduced to finding a regular solution in the upper semicircle for the equation

$$\Delta\Delta w = \frac{q}{D} \tag{28.1}$$

from the boundary conditions

$$w = \frac{dw}{dn} = 0 \quad \text{on } \gamma, \tag{28.2}$$

$$G(w) = H(w) = 0 \quad \text{on } l, \tag{28.3}$$

the operations G and H being defined by formulas (1.23), and D denoting the flexural rigidity of the plate.

The general outline of the method of solution is as follows. The given semi-circle is supplemented to a complete circle fixed along the entire contour and bent by a normal load of the same intensity q. In this circle we introduce a generalized load $p(x)$ distributed along its diameter l ($-1 < x < 1$) and representing a complex function of x* which should be chosen so as to fulfil conditions (28.3).

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* See the author's paper [46]. The same problem in the case when the condition of rigid fixing of a part of the edge is replaced by the condition of simple support (for the same part) has been solved by D. F. Muster and M. A. Sadowsky [94] in a different way.

In other words, the solution $w(x, y)$ is represented as the sum of two functions:

$$w(x, y) = \tilde{w}(x, y) + w_0(x, y), \tag{28.4}$$

where $\tilde{w}(x, y)$ are the deflections in the circular plate fixed along the contour under the given load q, and $w_0(x, y)$ are the deflections in the same plate bent by the forces $p(x)$ distributed along the diameter l which are to be determined.

As is known, the function $\tilde{w}(x, y)$ is of the form

$$\tilde{w}(x, y) = \frac{q}{64D}\left[(x^2+y^2)^2 - 2(x^2+y^2) + 1\right]. \tag{28.5}$$

We proceed to find the function w_0 by using representation (1.4) for it:

$$w_0(x, y) = \bar{z}\varphi_0(z) + z\overline{\varphi_0(z)} + \chi_0(z) + \overline{\chi_0(z)}. \tag{28.6}$$

Beginning the solution of the problem, we determine analytic functions $\varphi(z)$, $\psi(z)$ in the unit circle with singularities of the form

$$\varphi(z) = p\ln(z - z_0) + \varphi_*(z), \quad \psi(z) = \bar{p}\ln(z - z_0) - \frac{p\bar{z}_0}{z - z_0} + \psi_*(z),$$

where p is a constant, z_0 is a fixed point within the region, φ_* and ψ_* are holomorphic functions, from the boundary condition

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = 0 \quad \text{when} \quad |t| = 1. \tag{28.7}$$

The functions φ and ψ can easily be found by a method of N. I. Muskhelishvili.** We have for them under usual normalization conditions

$$\left.\begin{aligned} \varphi(z) &= p\ln(z - z_0) - p\ln(1 - z\bar{z}_0) - \bar{p}\,\frac{z(z - z_0)}{1 - z\bar{z}_0} - \\ &\qquad\qquad - \frac{1}{2}(p\bar{z}_0 + \bar{p}z_0)\,z, \\ \psi(z) &= \bar{p}\ln(z - z_0) - \frac{p\bar{z}_0}{z - z_0} - \bar{p}\ln(1 - z\bar{z}_0) + \\ &\qquad\qquad + \frac{\bar{p}\bar{z}_0(z - z_0)}{(1 - z\bar{z}_0)^2} - \frac{p\bar{z}_0^2 - 2\bar{p} + \bar{p}z_0\bar{z}_0}{1 - z\bar{z}_0}. \end{aligned}\right\} \tag{28.8}$$

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* In some cases it seems sufficient to introduce a real (normal, for example) load as additional forces $p(x)$. In these cases $p(x)$ is, of course, a real function.

** The reference is to Method 1 (see the beginning of Chap. II).

We now seek, according to formulas (28.8), holomorphic functions φ_0, ψ_0 in the semi-circle in the form

$$\left.\begin{aligned}\varphi_0(z) = \frac{1}{16\pi D}\int_{-1}^{1}\Big\{p(t)\,[\ln(z-t)-\ln(1-zt)]-\\ -\overline{p(t)}\,\frac{z(z-t)}{1-zt}-\frac{1}{2}\,[p(t)+\overline{p(t)}]\,tz\Big\}\,dt,\end{aligned}\right\} \qquad (28.9)$$

$$\psi_0(z) = \frac{1}{16\pi D}\int_{-1}^{1}\Big\{\overline{p(t)}\,[\ln(z-t)-\ln(1-zt)]-\\ -p(t)\,\frac{t}{z-t}+L(p,\,z;\;t)\Big\}\,dt$$

with

$$L(p,\,z;\;t)=\frac{\overline{p(t)}\,t\,(z-t)}{(1-zt)^2}-\frac{[p(t)+\overline{p(t)}]\,t^2-2\overline{p(t)}}{1-zt},$$

where $p(t)$ is a new unknown complex function of x defined on the interval l.

It is easily seen that the functions introduced by formulas (28.9) are holomorphic in any finite region of the z plane not containing points of the real axis. More precisely, the functions $\varphi_0'(z)$, $\psi_0(z)$ are made up of Cauchy-type integrals over a finite and an infinite segment of the x axis which complement each other to a complete straight line. In the following discussion these functions will be considered exclusively in the closed upper semi-circle where by $\ln(z-t)$ will be meant, for a fixed t, the branch for which $0 \leqslant \operatorname{Im}\ln(z-t) \leqslant \pi$. Moreover, according to formula (28.7) functions (28.9) will satisfy, for any $p(x)$, the boundary condition (28.2) apart from a constant.

From the first formula of (28.9) we obtain by differentiation

$$\varphi_0'(z)=\frac{1}{16\pi D}\int_{-1}^{1}\Big\{\frac{p(t)}{z-t}-zL(p,\,z;\;t)+\frac{1}{2}\,[p(t)+\overline{p(t)}]\,t\Big\}\,dt. \qquad (28.10)$$

If G_0, H_0 are operations formed for the biharmonic function w_0, and $\varphi_0(t)$, $\psi_0(t)$ are holomorphic functions corresponding to it by formula (28.6), then, on the basis of formula (1.25) due to S. G. Lekhnitskii and I. N. Vekua, we have on l

$$G_0(x)+i\int^{x}H_0(t)\,dt=-2\,(1-\nu)\,\frac{d}{dx}\,[-\varkappa\varphi_0(x)+x\overline{\varphi_0'(x)}+\overline{\psi_0(x)}], \qquad (28.11)$$

where $\varphi_0(x)$, $\varphi_0'(x)$, $\psi_0(x)$ denote the boundary values on l of the functions $\varphi_0(z)$, $\varphi_0'(z)$, $\psi_0(t)$ from the upper semi-circle.

With the use of representations (28.9) and (28.10) we now find

$$\Omega(p;\, x) \equiv -\varkappa\varphi_0(x) + x\overline{\varphi_0'(x)} + \overline{\psi_0(x)} =$$

$$= \frac{1}{16\pi D}\int_{-1}^{1}\Big\{\Big[-\varkappa\ln(x-t) + \overline{\ln(x-t)} + (\varkappa-1)\ln(1-xt) +$$

$$+ \frac{(t^2-1)(x^2-1)}{1-xt} + \frac{(t^2-1)(x^2-1)}{(1-xt)^2}\Big]p(t) +$$

$$+ \Big[\varkappa + (\varkappa + t^2)\frac{x^2-1}{1-xt}\Big]\overline{p(t)} + \frac{\varkappa+1}{2}\,[p(t) + \overline{p(t)}]\,tx\Big\}\,dt. \qquad (28.12)$$

On the other hand, for the operations $\widetilde{G}$ and $\widetilde{H}$ corresponding to the function $\widetilde{w}$ we have on the same interval

$$\widetilde{G}(x) = \frac{q}{16D}\,[(3\nu+1)\,x^2 - 1 - \nu], \qquad \widetilde{H}(x) = 0. \qquad (28.13)$$

By forming now the overall quantities $G = \widetilde{G} + G_0$, $H = \widetilde{H} + H_0$ for the required solution (28.4) and expressing the boundary conditions (28.3), we obtain an equation for the determination of $p(x)$:

$$2(1-\nu)\frac{d}{dx}\Omega(p;\, x) = \widetilde{G}(x), \qquad (28.14)$$

where Ω is given by formula (28.12).

Suppose that the function $p(x)$ satisfies the Hölder condition on l. Then, as is known, the following equalities are valid for any x from this interval:

$$\left.\begin{aligned}\frac{d}{dx}\int_{-1}^{1}\ln(x-t)\,p(t)\,dt &= -\pi i p(x) - \int_{-1}^{1}\frac{p(t)\,dt}{t-x},\\ \frac{d}{dx}\int_{-1}^{1}\overline{\ln(x-t)}\,p(t)\,dt &= \pi i p(x) - \int_{-1}^{1}\frac{p(t)\,dt}{t-x}.\end{aligned}\right\} \qquad (28.15)$$

Taking into account the above formulas, we perform the differentiation in equalities (28.14). Then, after some simplification we obtain an equation for $p(x)$:

$$\lambda p(x) + \frac{1}{\pi i}\int_{-1}^{1}\frac{p(t)\,dt}{t-x} +$$

$$+ \frac{1}{\pi i}\int_{-1}^{1}[n_0(x,\, t)\,p(t) + k_0(x,\, t)\,\overline{p(t)}]\,dt = f_0(x) \text{ on } l. \qquad (28.16)$$

Here

$$\left.\begin{aligned} n_0(x, t) &= -\frac{1}{1-xt} + \frac{\eta}{2}(t^2-1)\left[\frac{x}{1-xt} + \frac{x-t}{(1-xt)^2} + \right. \\ &\qquad\qquad\qquad \left. + \frac{2(x-t)}{(1-xt)^3}\right] + \frac{\lambda}{2}t, \\ k_0(x, t) &= \frac{\eta}{2}(\varkappa + t^2)\left[\frac{x}{1-xt} + \frac{x-t}{(1-xt)^2}\right] + \frac{\lambda}{2}t, \\ f_0(x) &= \frac{q}{4(1+\nu)i}\left[(3\nu+1)x^2 - 1 - \nu\right], \quad \lambda = \frac{2}{1-\nu}, \quad \eta = \frac{1-\nu}{1+\nu}. \end{aligned}\right\} \tag{28.17}$$

Equation (28.16) is a singular integral equation with Cauchy-type kernels.

According to the general theory of singular integral equations, the solution of Eq. (28.16) is to be sought in the class of functions which satisfy the Hölder condition on any closed part of the interval (—1, 1) and admit a singularity of order less than unity at $x = \pm 1$.* It should also be borne in mind that the functions $n_0(x, t)$ and $k_0(x, t)$, in contrast to the usual situation, are continuous functions of their arguments only on the interval (—1, 1); when $x = t = \pm 1$ these functions cease to be defined, and consequently the kernel of Eq. (28.16) admits, apart from a movable singularity inherent in the Cauchy kernel, fixed singularities at the ends of the interval.

Integral equations of this kind find application in boundary value problems of the theory of equations of the mixed type (see, for example, A. V. Bitsadze [7]). We think it is rather interesting to investigate the general properties of such equations, and in particular to examine the question of their solvability in a particular class of functions. A detailed study of Eq. (28.16) would decide, for example, whether a rigorous solution of our problem is possible in terms of potentials (28.9). Below we shall be concerned with an approximate solution of Eq. (28.16), assuming, from physical considerations, that this equation admits a solution $p(x)$ continuous in Hölder's sense on the closed interval l. The additional condition which is introduced here regarding the unknown $p(x)$ requires, in effect, that $p(x)$ be bounded near the ends of the interval.**

The last requirement is equivalent to a definite hypothesis about the behaviour of the characteristic physical quantities in the closed semi-circle. Under the above assumptions regarding $p(x)$, the first derivatives of w, as is clear from formulas (28.9), (28.10), are continuous in the closed region, and hence the conditions which guarantee the uniqueness of solution of the boundary value problem are fulfilled (see Sec. 3).

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* Stated otherwise, the solution of Eq. (28.16) is sought in the class H^* on l (for the definition of the class H^* see Sec. 3, Par. 1°)

** Strictly speaking, there is no need for such a limitation; the problem can be solved just as well without it. We introduce it to simplify the numerical algorithm.

As to the above-mentioned physical quantities (the bending and twisting moments G_x, G_y, H_{xy}), they are continuous in the closed semi-circle with the excluded angular points in the neighbourhood of which the following estimate is valid (see, for example, G. N. Polozhii [103]):

$$|G_x|,\quad |G_y|,\quad |H_{xy}| < A \ln \frac{1}{|z \pm 1|} \quad (A = \text{constant}). \tag{28.18}$$

Under these conditions it may be stated that the solution of Eq. (28.16) (if it exists) is unique.

We set $p(x) = \alpha(x) + i\beta(x)$ in Eq. (28.16) and separate the real and imaginary parts. For the determination of α and β we then obtain a system of two equations:

$$\left.\begin{aligned} &\frac{\lambda}{2}\alpha(x) + \frac{1}{2\pi}\int_{-1}^{1}\frac{\beta(t)\,dt}{t-x} + \frac{1}{2\pi}\int_{-1}^{1} n(x, t)\,\beta(t)\,dt = 0, \\ &\frac{\lambda}{2}\beta(x) - \frac{1}{2\pi}\int_{-1}^{1}\frac{\alpha(t)\,dt}{t-x} + \frac{1}{2\pi}\int_{-1}^{1} k(x, t)\,\alpha(t)\,dt = f(x) \text{ on } l, \end{aligned}\right\} \tag{28.19}$$

where

$$\left.\begin{aligned} n(x, t) &= -\frac{1}{1-xt}\left[t + \lambda x + \frac{\lambda(x-t)}{1-xt} - \eta\frac{(x-t)(t^2-1)}{(1-xt)^2}\right], \\ k(x, t) &= -\frac{1}{1-xt}\left[-t + (1+\eta t^2)x + \right. \\ &\qquad \left. + \frac{(1+\eta t^2)(x-t)}{1-xt} + \eta\frac{(x-t)(t^2-1)}{(1-xt)^2}\right] - \lambda t, \\ &\qquad f(x) = \frac{1}{2i} f_0(x). \end{aligned}\right\} \tag{28.20}$$

By using formulas (3.4) and (3.5) (Chap. I), it can easily be shown that, if $\alpha(x)$, $\beta(x)$ is a solution of system (28.19) which satisfies the Hölder condition on l including the ends, then the function $\beta(x)$ must vanish at the ends of the interval

$$\beta(1) = \beta(-1) = 0.$$

Furthermore, it is proved that the function $\beta(x)$ is even in x and $\alpha(x)$ is odd:

$$\beta(x) = \beta(-x), \qquad \alpha(x) = -\alpha(-x) \text{ on } l.$$

The foregoing properties of α and β will be used below in constructing a numerical solution of the system of equations (28.19).

After the function $p(x)$ is found, formulas (28.9) give the unknown holomorphic functions $\varphi_0(z)$, $\psi_0(z)$. As to the function $\chi_0(z)$ appearing in the expression for the deflection, it is represented by

$$\chi_0(z) = \chi_*(z) + C,$$

where χ_* is a primitive of ψ_0 and C is a real constant.

To determine C, it is sufficient to require the fulfilment of the first of conditions (28.2) at any one point of the arc γ, say at the point $t = i$,

$$\operatorname{Re}\{\bar{t}\varphi_0(t) + \chi_*(t)\} = -C \quad \text{at } t = i. \tag{28.21}$$

On the basis of representations (1.20) all unknowns characterizing the state of stress in the plate may be determined in terms of the functions $\varphi_0(z)$, $\psi_0(z)$. For example, the bending moment M ($M = -DG$) along the arc γ is expressed by the formula

$$M = -\frac{q}{8} + 2D(1-\nu)\operatorname{Re}\left\{\frac{d}{dt}\left[-\varkappa\varphi_0(t) + t\overline{\varphi_0'(t)} + \overline{\psi_0(t)}\right]\right\}$$

which, after simple computations, may be put in the form

$$-M = \frac{q}{8} + \frac{1}{2\pi}\operatorname{Re}\int_{-1}^{1}\left\{\frac{p(\sigma)}{t-\sigma} + \frac{\sigma[p(\sigma) - \overline{p(\sigma)}]}{1-\sigma t} - \right.$$

$$\left. - \frac{\overline{p(\sigma)}\,(t-\sigma)}{1-\sigma t}\left[1 + \frac{1}{1-\sigma t}\right] - \frac{1}{2}[p(\sigma) + \overline{p(\sigma)}]\,\sigma\right\} d\sigma \quad \text{on } \gamma. \tag{28.22}$$

The bending moment along the diameter of the plate which must vanish according to the condition of the problem is

$$-M = \frac{q}{16}[(3\nu+1)x^2 - 1 - \nu] + \frac{1}{2}\left[\beta(x) - \frac{1+\nu}{2\pi}\int_{-1}^{1}\frac{\alpha(t)\,dt}{t-x} + \right.$$

$$\left. + \frac{1+\nu}{2\pi}\int_{-1}^{1}k(x, t)\,\alpha(t)\,dt\right] \quad \text{on } l. \tag{28.23}$$

Finally, for the deflections w we have

$$w(x, y) = \tilde{w}(x, y) + 2\operatorname{Re}[\bar{z}\varphi_0(z) + \chi_0(z)]. \tag{28.24}$$

The approximate solution of system (28.19) will be found by the procedure indicated in Sec. 13. Seeking a solution bounded at both ends of the interval, we approximate the unknown function $p(x)$ by a trigonometric interpolation polynomial in ϑ ($x = \cos\vartheta$) constructed with the nodes

$$x_m = \cos\vartheta_m, \qquad \vartheta_m = \frac{m\pi}{n+1} \qquad (m = 1, 2, \ldots, n)$$

of the form (see Sec. 13, Par. 3°)

$$L_n[p; x] = \frac{2}{n+1}\sum_{k=1}^{n} p_k \sum_{m=1}^{n}\sin m\vartheta_k \sin m\vartheta, \tag{28.25}$$

where

$$p_k = p(x_k) = \alpha(x_k) + i\beta(x_k)$$

(n is a natural number). This polynomial as a function of x is of the form $\sqrt{1-x^2}\,P_{n-1}(x)$, where $P_{n-1}(x)$ is a polynomial in x with complex coefficients of degree not greater than $n-1$.

We then obtain a system of linear equations in α_k, β_k of the form

$$\left.\begin{array}{c}\dfrac{\alpha_m}{\nu+1}-\sum\limits_{k=1}^{n} a_{mk}\beta_k=0,\\[2ex] \dfrac{\beta_m}{\nu+1}+\sum\limits_{k=1}^{n} b_{mk}\alpha_k=f_m\\[2ex] (m=1,\ 2,\ \ldots,\ n),\end{array}\right\} \tag{28.26}$$

where

$$a_{mk}=\frac{\sin\vartheta_k}{n+1}\left[\frac{\varepsilon_{mk}}{\cos\vartheta_m-\cos\vartheta_k}-\frac{1}{2}n(\cos\vartheta_m,\ \cos\vartheta_k)\right],$$

$$b_{mk}=\frac{\sin\vartheta_k}{n+1}\left[\frac{\varepsilon_{mk}}{\cos\vartheta_m-\cos\vartheta_k}+\frac{1}{2}k(\cos\vartheta_m,\ \cos\vartheta_k)\right],$$

$$f_m=-\frac{q}{8(1+\nu)}\left[(3\nu+1)\cos^2\vartheta_m-1-\nu\right],$$

$$\varepsilon_{mk}=1 \text{ when } |m-k|=1,\ 3,\ \ldots,\ \varepsilon_{mk}=$$
$$=0, \text{ when } |m-k|=0,\ 2,\ \ldots\ .$$

The solution of system (28.26) determines an approximate solution of our boundary value problem. The physical quantities are then defined in the form of simple integrals to the evaluation of which, according to formula (28.25), it is advisable to apply the Gauss-type formula

$$\frac{1}{\pi}\int\limits_{-1}^{1}\sqrt{1-t^2}\,F(t)\,dt=\frac{1}{n+1}\sum_{k=1}^{n}\sin^2\vartheta_k F(\cos\vartheta_k). \tag{28.27}$$

System (28.26), as seen from the construction of the coefficients a_{mk}, b_{mk}, has a rather simple structure, and in consequence it is not at all difficult to find its solution for small n. However, in view of the above-mentioned singularities of the kernels $n(x, t)$ and $k(x, t)$, to obtain a solution possessing a high accuracy, it is apparently necessary to solve this system for large values of n, which requires the use of a computer.

Let us consider, as one of the approximations, a solution for $n=8$. On the basis of the above-mentioned properties of α and β we have a system of eight equations. The nodal values of the unknown functions are found to be

$$\left.\begin{aligned} &\alpha_1 = 0.05526, && \beta_1 = 0.00139, \\ &\alpha_2 = 0.04005, && \beta_2 = 0.01194, \\ &\alpha_3 = 0.02458, && \beta_3 = 0.01792, \\ &\alpha_4 = 0.00969, && \beta_4 = 0.02569, \\ \alpha_k = -\alpha_{9-k}, \quad & \beta_k = \beta_{9-k}, && k = 5,\ 6,\ 7,\ 8. \end{aligned}\right\} \tag{28.28}$$

In order to get an idea of the accuracy of the approximate solution $p(x)$ defined by values (28.28), we calculate the bending moment M at the point $x = y = 0$ located at the middle of the interval between two adjacent nodes. By applying the quadrature formula (28.27) to the integral on the right-hand side of relation (28.23) and substituting values (28.28), we find

$$M\,|_{x=y=0} = 0.00529q. \tag{28.29}$$

The bending moment along the arc γ defined by formula (28.22) has a maximum value at the point $t = i$. Substituting $t = i$ in this formula and evaluating the integral as before, we obtain

$$M\,|_{t=i} = -\frac{q}{8} - \frac{1}{2\pi}\int_{-1}^{1}\left\{\left[\frac{4}{(t^2+1)^2} - 1\right] t\alpha(t) - \frac{(t^2-1)^2}{(t^2+1)^2}\,\beta(t)\right\} dt =$$
$$= -0.13822q. \tag{28.30}$$

We further consider the deflection at the middle of the diameter. In order to find deflections, it is necessary, by the above, to determine the function

$$\chi_0(z) = \int \psi_0(z)\,dz + C,$$

where the arbitrary real constant C must be found from condition (28.21). On the basis of formulas (28.9) and (28.21) we have after elementary computations

$$\chi_0(z) = \frac{1}{16\pi D}\int_{-1}^{1}\left\{\left[z\overline{p(t)} - t\overline{p(t)} - tp(t)\right]\left[\ln(z-t) - \ln(1-tz)\right] + \right.$$
$$\left. + \overline{p(t)}\,\frac{1-t^2}{t(1-tz)} + \left[p(t) + \overline{p(t)}\right]\frac{t^2-1}{2t}\right\} dt. \tag{28.31}$$

Formula (28.24), with (28.31), now gives

$$w(0,\ 0) = \tilde{w}(0,\ 0) + 2\,\mathrm{Re}\,\chi_0(0) = \frac{q}{64D} - \frac{1}{4\pi D}\int_{-1}^{1} t\alpha(t)\ln|t|\,dt.$$

By applying the quadrature formula to the evaluation of the integral and using table (28.28), we find*

$$w(0,\ 0) = 0.01663q/D. \tag{28.32}$$

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* In the above-cited paper of the author [46] there are misprints in the equalities defining the numerical values of M (for $t = i$) and $w(0,\ 0)$.

29. Mixed Problems for a Circle

The method for the solution of the mixed problem developed in Sec. 23 is appreciably simplified in the case of a complete circle and draws, in effect, on the functional equation of the problem corresponding to the conditions of the mixed type. Before explaining this statement, it might be well to recall the formulation of the fundamental problems pertaining, say, to the case of the bending of thin plates. The finite and singly connected region of the middle surface is denoted by S and its boundary, a simple closed contour, by L.

1°. As throughout, use is made of formula (1.21) giving the general representation of solutions of the bending equation (1.19)

$$w = 2\,\mathrm{Re}\,[\bar{z}\varphi(z) + \chi(z)] + w_0(z, \bar{z}), \tag{29.1}$$

where w_0 is a particular solution of the non-homogeneous equation.

In Problem I when the plate is fixed along the entire edge, for the determination of the functions φ and ψ we have the boundary condition (3.9):

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = f(t) \text{ on } L, \tag{29.2}$$

$$f(t) = -\frac{\partial w_0}{\partial \bar{t}}. \tag{29.3}$$

In the case of Problem II the boundary condition, which expresses the absence of external forces at the edge, is of the form [formula (3.12)]

$$-\varkappa\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = f(t) + iC_1 t + \delta \quad \text{on } L, \tag{29.4}$$

where $f(t)$ is a given single-valued continuous function on the boundary of the region defined by the formula

$$f(t) = \frac{1}{2(1-\nu)} \int^{t} \left[G(w_0) + i\int^{s} H(w_0)\,d\tau\right] dt, \quad \varkappa = \frac{3+\nu}{1-\nu}, \tag{29.5}$$

C_1 is an arbitrary real constant and δ is an arbitrary complex constant; the operators G and H are given by formulas (1.23).

Finally, the boundary condition of Problem III when the edge of a plate is simply supported may also be expressed, under the additional condition that the curvature of the contour does not become zero, in the form of one complex equality of (6.1):

$$\lambda[\varphi'(t) + \overline{\varphi'(t)}] + \bar{t}''[\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)}] = f(t) \text{ on } L \tag{29.6}$$

and

$$f(t)=\frac{1}{2}\left[h_1+\frac{\partial^2 h_2}{\partial s^2}-\frac{i}{\rho}\frac{\partial h_2}{\partial s}\right], \qquad t''=\frac{d^2t}{ds^2}, \tag{29.7}$$

$$h_1=-\frac{G(w_0)}{1-\nu}, \quad h_2=-w_0(t,\bar{t}), \quad \lambda=\frac{2}{1-\nu}; \tag{29.8}$$

ρ is the radius of curvature of the contour.

It might also be well to recall the degree of determinacy of the complex potentials.

With moments and forces given throughout the body, the functions $\varphi(z)$ and $\psi(z)$ are determinable apart from terms of the form

$$\varphi(z)=iCz+\gamma, \qquad \psi(z)=\gamma', \tag{29.9}$$

where C is an arbitrary real constant and γ, γ' are arbitrary complex constants. From this it follows that in the case of Problem II when static conditions are prescribed on the boundary the functions φ and ψ may be subjected to further conditions (assuming the origin to be in the region S)

$$\operatorname{Im}\varphi'(0)=0, \qquad \varphi(0)=0, \qquad \psi(0)=0. \tag{29.10}$$

Under these conditions the complex potentials, and with them the constants C_1 and δ appearing on the right-hand side of (29.4), are determined uniquely. On the same basis we can assign beforehand to the just mentioned constants certain specific values,* but then, instead of (29.10), there will be left only one condition

$$\varphi(0)=0 \quad \text{or} \quad \psi(0)=0.$$

In Problems I and III values of deflections are given on the boundary L and the right-hand sides of (29.2) and (29.6) are completely determined. The normalization conditions in these cases are of the form

$$\operatorname{Im}\varphi'(0)=0, \qquad \varphi(0)=0 \quad \text{or} \quad \psi(0)=0.$$

The above conditions leave no arbitrariness whatsoever in the functions φ and ψ defining the solutions of Problems I and III.

We now turn to the case of a circle. For simplicity, suppose that the complete boundary γ of the unit circle centred at the origin is divided into two equal parts, namely the upper and lower semi-circumferences γ^+ and γ^-. We shall prescribe conditions of one type over the part γ^+ and conditions of another type over γ^-. Three cases are possible.

First case. The semi-circumference γ^+ is free from forces and the other one is clamped. According to (29.2) and (29.4) the boundary conditions are written as*

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* This is equivalent to implying by the right-hand side of (29.5) a definite primitive.

$$\left.\begin{aligned}&\varphi(t)+\overline{t\varphi'(t)}+\overline{\psi(t)}-(\varkappa+1)\varphi(t)=f(t) \text{ on } \gamma^{+},\\ &\varphi(t)+\overline{t\varphi'(t)}+\overline{\psi(t)}=f(t) \text{ on } \gamma^{-}, \quad \psi(0)=0.\end{aligned}\right\} \quad (29.11)$$

Second case. A circular plate is simply supported along γ^{+} and clamped along γ^{-}. In this case, according to equalities (29.2) and (29.6), after multiplying the second of these by $-t$, we arrive at the boundary value problem

$$\left.\begin{aligned}&\varphi(t)+\overline{t\varphi'(t)}+\overline{\psi(t)}-\lambda t\,[\varphi'(t)+\overline{\varphi'(t)}]=-tf(t) \text{ on } \gamma^{+},\\ &\varphi(t)+\overline{t\varphi'(t)}+\overline{\psi(t)}=f(t) \text{ on } \gamma^{-},\end{aligned}\right\} \quad (29.12)$$

$$\operatorname{Im}\varphi'(0)=0, \qquad \psi(0)=0.$$

Third case. The upper part of the boundary is simply supported and the lower is free. The conditions on the boundary of the circle are of the form

$$\left.\begin{aligned}&-\varkappa\varphi(t)+\overline{t\varphi'(t)}+\overline{\psi(t)}+(\varkappa+1)\varphi(t)-\\ &-\lambda t\,[\varphi'(t)+\overline{\varphi'(t)}]=-tf(t) \text{ on } \gamma^{+},\\ &-\varkappa\varphi(t)+\overline{t\varphi'(t)}+\overline{\psi(t)}=f(t) \text{ on } \gamma^{-}, \qquad \psi(0)=0.\end{aligned}\right\} \quad (29.13)$$

We now perform the following operation on both sides of the above equalities:

$$\frac{1}{2\pi i}\int\limits_{\gamma}\frac{dt}{t-z}, \qquad |z|<1. \quad (29.14)$$

Assume in the unit circle

$$\varphi(z)=\sum_{n=0}^{\infty}a_n z^n \quad (29.15)$$

and take into account the corresponding normalization conditions. As a result we come to the following functional equations:

in the first case

$$\varphi(z)+\overline{a}_1 z+2\overline{a}_2-\frac{\varkappa+1}{2\pi i}\int\limits_{\gamma^{+}}\frac{\varphi(t)\,dt}{t-z}=A(z), \quad (29.16)$$

$$A(z)=\frac{1}{2\pi i}\int\limits_{\gamma}\frac{f(t)\,dt}{t-z}; \quad (29.17)$$

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* The solution of this "fundamental mixed problem" is constructed in closed form under rather general conditions (see, for example, N. I. Muskhelishvili [93]).

in the second case

$$\varphi(z)+\bar{a}_1 z+2\bar{a}_2-\frac{\lambda}{2\pi i}\int_{\gamma^+}\frac{t\,[\varphi'(t)+\overline{\varphi'(t)}]}{t-z}\,dt=B(z), \tag{29.18}$$

$$B(z)=\frac{1}{2\pi i}\int_{\gamma}\frac{g(t)\,dt}{t-z}, \qquad g(t)=-tf(t) \text{ on } \gamma^+,$$

$$g(t)=f(t) \text{ on } \gamma^-; \tag{29.19}$$

in the third case

$$-\varkappa\varphi(z)+\bar{a}_1 z+2\bar{a}_2+\frac{1}{2\pi i}\int_{\gamma^+}\frac{M[\varphi;\,t]\,dt}{t-z}=B(z), \tag{29.20}$$

$$M[\varphi;\,t]=(\varkappa+1)\,\varphi(t)-\lambda t\,[\varphi'(t)+\bar{\varphi}'(t)]$$

and $B(z)$ is of the same form as (29.19). The functions $f(t)$ and $g(t)$ on the right-hand sides under the Cauchy-type integrals have discontinuities of the first kind at the points $t=\pm 1$.

The above functional equations, after expanding their both sides in Taylor's series and subsequently comparing the coefficients of z^n, lead to infinite systems of linear algebraic equations whence all unknowns a_n must be determined uniquely. Having determined $\varphi(z)$, we can find, with the use of the Cauchy integral, the other unknown function, $\psi(z)$, directly from the boundary condition of the problem.

Let us discuss in greater detail the second of the above cases.*

2°. In this case, as will be shown later, one of the unknown functions of a complex argument may be expressed in terms of the other in elementary form, and this allows a reduction of the mixed problem to a boundary value problem for one analytic function. Moreover, the reduced mixed problem leads to a functional equation somewhat more convenient to deal with than (29.18).

Revert to the boundary conditions (29.12) and rewrite them in the new notation (we write φ_1, ψ_1, χ_1 for φ, ψ, χ):

$$\left.\begin{aligned}&\lambda\,[\varphi_1'(t)+\overline{\varphi_1'(t)}]-\bar{t}\,[\varphi_1(t)+t\overline{\varphi_1'(t)}+\overline{\psi_1(t)}]=f(t) \text{ on } \gamma^+,\\&\varphi_1(t)+t\overline{\varphi_1'(t)}+\overline{\psi_1(t)}=f(t) \text{ on } \gamma^-,\end{aligned}\right\} \tag{29.21}$$

$$\operatorname{Im}\varphi_1'(0)=0, \qquad \varphi_1(0)=0. \tag{29.22}$$

The above condition written in complex form includes, of course, the condition which specifies deflections along the entire contour of the circle

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* The solution of mixed problems for a circle is given in D. I. Sherman's papers [125, 126]. The method of solution proposed below (pertaining to the second case) is somewhat different from that indicated in the first of the aforenamed papers. It appears to be more elementary and leads to an infinite system of a very simple structure. In the meantime, D. I. Sherman's method, in contrast to the one proposed here, leads to regular systems.

$$w = 0 \quad \text{on } \gamma \tag{29.23}$$

which, on the basis of (29.1), we represent as

$$\bar t\varphi_1(t) + t\overline{\varphi_1(t)} + \chi_1(t) + \overline{\chi_1(t)} = -w_0(t, \bar t) \quad \text{on } \gamma. \tag{29.24}$$

Suppose that the intensity of normal load is constant and take a particular solution w_0 in the form

$$w_0(z, \bar z) = \frac{q}{64D} z^2\bar z^2 \qquad (q = \text{constant}). \tag{29.25}$$

Using (29.25) and the normalization conditions for $\varphi_1(z)$ we perform operation (29.14) on both sides of equality (29.24). We obtain

$$\frac{\varphi_1(z)}{z} + \chi_1(z) = \text{constant}. \tag{29.26}$$

The constant on the right-hand side is easily chosen by requiring the fulfilment of condition (29.24) at any one point of the circle. After determining this constant, we insert the function $\chi_1(t)$ from (29.26) in (29.1) and represent the unknown deflections $w(x, y)$ as

$$w = 2\,\mathrm{Re}\left\{\left(\bar z - \frac{1}{z}\right)\varphi_1(z)\right\} + W(z, \bar z), \tag{29.27}$$

$$W(z, \bar z) = \frac{q}{64D}(z^2\bar z^2 - 1).$$

To determine $\varphi_1(z)$, we turn to condition (29.21). We substitute in these equalities the values of the function $\psi_1(t)$ determined from (29.26), multiply the first equality by $1 - \nu$, the second by $(1 + \nu)\,\bar t$ and introduce a new unknown function

$$\varphi(z) = \frac{\varphi_1(z)}{z} \tag{29.28}$$

also holomorphic inside the circle. For $\varphi(z)$ we then obtain a (real) boundary condition given by

$$\left.\begin{aligned}(1+\nu)[\varphi(t)+\overline{\varphi(t)}]+2[t\varphi'(t)+t\overline{\varphi'(t)}] &= A+B \text{ on } \gamma^+,\\ (1+\nu)[\varphi(t)+\overline{\varphi(t)}] &= A \text{ on } \gamma^-.\end{aligned}\right\} \tag{29.29}$$

The quantities A and B are easily calculated by formulas (29.3), (29.7), (29.8) and (29.25). Indeed,

$$A = -(1+\nu)\,\bar t\,\frac{\partial w_0}{\partial \bar t}, \qquad A + B = -\frac{G(w_0)}{2}. \tag{29.30}$$

To calculate $A + B$, it is sufficient to recall the well-known identity which is valid along any contour with continuous curvature

$$G = \Delta - \frac{1-\nu}{\rho}\frac{\partial}{\partial n} \qquad \left(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right),$$

where ρ is the radius of curvature, n is the outward normal to L. In our case of the unit circle we have therefrom

$$-G(w_0) = -\Delta w_0 + (1-\nu)\frac{\partial w_0}{\partial r}.$$

Equalities (29.30), with (29.25), give finally

$$A = -\frac{1+\nu}{32}\frac{q}{D}, \qquad B = -\frac{1}{16}\frac{q}{D}. \tag{29.31}$$

Performing now operation (29.14) on equalities (29.29), we arrive at the required functional equation

$$(1+\nu)[\varphi(z)+a_0] + \frac{1}{\pi i}\int\limits_{\gamma^+}\frac{t\varphi'(t)+\overline{t\varphi'(t)}}{t-z}\,dt = A + g(z), \tag{29.32}$$

$$g(z) = \frac{B}{2\pi i}\int\limits_{\gamma^+}\frac{dt}{t-z}.$$

By virtue of the first of conditions (29.22) the constant $a_0 = \varphi(0)$ is real.

Assuming for $|z| < 1$

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n, \tag{29.33}$$

we represent Eq. (29.32) in the form

$$2(1+\nu)a_0 + (1+\nu)\sum_{n=1}^{\infty} a_n z^n + \sum_{n=0}^{\infty}\Omega_n z^n = A + \sum_{n=0}^{\infty} g_n z^n, \tag{29.34}$$

$$\Omega_n = \frac{1}{\pi i}\int\limits_{\gamma^+}\Omega(t)\,t^{-n-1}\,dt, \qquad g_n = \frac{B}{2\pi i}\int\limits_{\gamma^+} t^{-n-1}\,dt, \tag{29.35}$$

$$\Omega(t) = \sum_{k=1}^{\infty} k\,[a_k t^k + \bar{a}_k t^{-k}] \text{ on } \gamma^+. \tag{29.36}$$

Supposing the legitimacy of termwise integration along γ^+ of series (29.36), we use formulas (23.28) and compare the coefficients of z^n on both sides of equality (29.34). As a result we come to the system of equations

$$\left.\begin{aligned} &2(1+\nu)a_0 + \frac{1}{\pi i}\sum_{k=1}^{\infty} k\delta_k(a_k - \bar{a}_k) = A + \frac{B}{2},\\ &(1+\nu+n)a_n + \frac{1}{\pi i}\sum_{k=1}^{\infty}{}' k\delta_{k-n}a_k - \frac{1}{\pi i}\sum_{k=1}^{\infty} k\delta_{k+n}\bar{a}_k =\\ &\qquad\qquad = -\frac{B}{2}\frac{\delta_n}{\pi i} \qquad (n \geqslant 1), \end{aligned}\right\} \tag{29.37}$$

$$\delta_m = \frac{(-1)^m - 1}{m}, \qquad m = \pm 1, \pm 2, \ldots .$$

A prime on the symbol Σ denotes that the value $k = n$ is omitted in the summation.

We introduce, for convenience, new unknowns

$$2a_0 = \alpha_0, \qquad ka_k = \alpha_k + i\beta_k, \qquad k \geqslant 1$$

and separate the real and imaginary parts in (29.37). The infinite system in α_n, β_n then becomes

$$\left.\begin{aligned} &(1+\nu)\,\alpha_0+\frac{2}{\pi}\sum_{k=1}^{\infty}\delta_k\beta_k=A+\frac{B}{2},\\ &\left(\frac{1+\nu}{n}+1\right)\alpha_n+\frac{1}{\pi}\sum_{k=1}^{\infty}{}'\,(\delta_{k-n}+\delta_{k+n})\,\beta_k=0,\\ &\left(\frac{1+\nu}{n}+1\right)\beta_n+\frac{1}{\pi}\sum_{k=1}^{\infty}{}'\,(\delta_{k+n}-\delta_{k-n})\,\alpha_k=\frac{B}{2\pi}\delta_n \quad (n\geqslant 1).\end{aligned}\right\} \tag{29.38}$$

The analysis of system (29.38) will not be taken up here. Suppose that it has a solution suitable for the solution of our mixed problem. Then, because of the fact that $\delta_{2n} = 0$ when $n \geqslant 1$, we conclude that $\alpha_{2n-1} = \beta_{2n} = 0$ for all $n \geqslant 1$. On the same basis each unknown α_{2n} separately is expressed in terms of the unknowns β_{2n-1} and vice versa. It is also easily verified that

$$|\,\delta_{k-n} + \delta_{k+n}\,| < \pi, \qquad |\,\delta_{k-n} - \delta_{k+n}\,| < \pi$$

for any k and n, $k \neq n$.

In view of the foregoing properties of system (29.38) the coefficients of the diagonal terms in each of its equations exceed all the others and it is therefore advisable to solve the truncated system by the iteration method if, of course, it converges.

After determining the solution of (29.38), the unknown function $w(x, y)$, according to (29.27), is represented as

$$w(x, y) = 2\,\mathrm{Re}\,\{(z\bar{z} - 1)\,\varphi(z)\} + W(z, \bar{z}).$$

II. THE PROBLEM OF THE REINFORCING OF PLATES

Under conditions of generalized plane stress, consider the problem of stresses in a plate reinforced with stiffeners. By a stiffener is meant a curved elastic bar of uniform section (in general, of different material) which is continuously attached to the plate along its edge or along any other line located partly or wholly within the physical region.

Plates with stiffeners find wide application in engineering practice, particularly in aircraft construction. A study of forces transferred to the plate through a stiffener has been given due attention by elasticists; a great many works of Soviet and foreign authors have been devoted to this subject (for example, E. Melan [79], E. L. Buell [15], J. L. Sanders, Jr. [106], E. H. Brown [11], W. T. Koiter [59], R. Greif and J. L. Sanders, Jr. [32], M. P. Sheremet'ev [117], G. N. Savin and N. P. Fleishman [108], N. Kh. Arutyunyan [4], N. Kh. Arutyunyan and S. M. Mkhitaryan [5], N. Kh. Arutunyan and S. M. Mkhitaryan [2], etc.).

The construction of an effective solution to the problem in its general formulation presents great difficulties. It is therefore often necessary to consider an idealized model of the real structure which leads to a considerable simplification of the corresponding mathematical problem.

This part is concerned with problems involving straight stiffeners, semi-infinite and infinite, as well as finite ones.

It should be noted that these problems have been considered in the work of many authors.* They have proposed various methods of approximate solution which enable one to clarify the qualitative nature of the stresses in particular parts of a structure and at the same time to carry out the necessary numerical calculations.

Our approach is fundamentally different from those employed previously and makes it possible, in the case of a semi-infinite stringer, to construct the solution of the problem in finite form (in terms of Fourier integrals). This is achieved by applying the techniques of Cauchy-type integrals in conjunction with the Wiener-Hopf method. In the case of an infinite stringer the plate may, by hypothesis, have an isolated crack and the problem is solved approximately.

30. Half-plane with a Semi-infinite Stiffener Along Its Boundary**

We place the elastic medium in the lower half-plane of the z plane and make the axis of the stringer (stiffener) coincide with the positive part of the real axis; the stringer has one end at the origin and extends to infinity with the other.

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* Reference will be made in the relevant places.

** This problem has been solved at different times in the above-mentioned works of E. L. Buell [15] and W. T. Koiter [59]. Its solution in finite form has been given in the author's paper [53] which is reproduced here with only slight modifications. W. T. Koiter [59] and also E. H. Brown [11] have, in addition, studied a somewhat different problem of the reinforcing of a plate (see Sec. 31).

It is assumed that the stiffener represents an elastic line of different material which can act only in tension.* Suppose that the stresses (in the plate and stringer) are produced by a single axial force p_0 applied at the end of the stringer $z = 0$ and directed along the x axis.

Let E and ν be the elastic constants of the plate, E_0 the modulus of elasticity of the bar, h the thickness of the plate, S_0 the cross section of the bar which is assumed to be uniform.

The part of the boundary of the half-plane to the left of the origin is not loaded, by hypothesis. The boundary conditions here are therefore

$$\sigma_y = \tau_{xy} = 0 \qquad (x < 0). \tag{30.1}$$

The boundary conditions over the remainder of the boundary where the plate is connected to the stringer are the conditions for the equilibrium of any part $(0, x)$ of the latter which are given (in the absence of the bending moment at sections) by the following two equalities:**

$$p_0 - h\int_0^x \tau_{xy}\,dt + k\sigma_x = 0, \qquad -h\int_0^x \sigma_y\,dt = 0 \qquad (x > 0), \tag{30.2}$$

where

$$k = \frac{E_0 S_0}{E}.$$

These two equalities together give

$$p_0 - h\int_0^x (\tau_{xy} + i\sigma_y)\,dt + k\sigma_x = 0 \qquad (x > 0). \tag{30.3}$$

We now use the familiar Kolosov-Muskhelishvili representations (1.2):

$$\sigma_x + \sigma_y = 2\,[\varphi'(z) + \overline{\varphi'(z)}], \qquad \sigma_y - \sigma_x + 2i\tau_{xy} = 2\,[\bar{z}\varphi''(z) + \psi'(z)] \tag{30.4}$$

and Muskhelishvili's formula (1.6)

$$-i\int_0^t (\tau_{xy} + i\sigma_y)\,d\tau = \varphi(t) + \overline{t\varphi'(t)} + \overline{\psi(t)} + \text{constant}. \tag{30.5}$$

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* With a more accurate model of a stringer when it also possesses flexural rigidity, the problem becomes much more complicated. On the other hand, as M. P. Sheremet'ev [116] has shown by considering a circular hole reinforced with an elastic stiffener, the flexural rigidity of the latter has but a slight influence on the picture of the state of stress as a whole. A more detailed study of the effect of the flexural rigidity of a stiffener was carried out in a recent paper by R. Muki and E. Sternberg [85] which is primarily devoted to the discussion of the well-known Melan solution (E. Melan [79]).

** The derivation of these equations is given below in a more general case (see the next section).

On the basis of these formulas the boundary conditions (30.1), (30.3) of our problem, omitting non-essential constants, are put in the form

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = 0 \qquad (t < 0), \tag{30.6}$$

$$ip_0 + h\,[\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)}] + \\ + ik\,\mathrm{Re}\,[\varphi'(t) + \overline{\varphi'(t)} - t\overline{\varphi''(t)} - \overline{\psi'(t)}] = 0 \qquad (t > 0). \tag{30.7}$$

From the physical significance of the problem it is apparent that the stresses are not bounded in the closed half-plane. At the point $z = 0$ where the plate is acted on by the load through the stringer, they may become infinite with degree less than unity. For large $|z|$, however, because of the absence of external forces in infinitely remote parts of the medium the stresses are vanishingly small.

Let us calculate the resultant of the external forces applied to the boundary of the half-plane. On the basis of formulas (30.1) and (30.2) we have

$$X = \int_{-\infty}^{\infty} \tau_{xy}\,dt = \frac{p_0}{h}, \qquad Y = \int_{-\infty}^{\infty} \sigma_y\,dt = 0. \tag{30.8}$$

According to the conceptions of N. I. Muskhelishvili, the presence of a non-zero resultant entails the unboundedness of the complex potentials $\varphi(z)$, $\psi(z)$ at infinity; for large $|z|$ these potentials are of the form (see N. I. Muskhelishvili [93], Sec. 90)

$$\varphi(z) = -\frac{X}{2\pi}\ln z + \varphi_0(z), \qquad \psi(z) = \frac{X}{2\pi}\ln z + \psi_0(z), \tag{30.9}$$

where $\varphi_0(z)$, $\psi_0(z)$ are holomorphic functions in the half-plane which admit the asymptotic representation

$$\varphi_0(z) = o(1) + \text{constant}, \quad \psi_0(z) = o(1) + \text{constant}. \tag{30.10}$$

According to our assumption as to the nature of the stresses in the neighbourhood of the point $z = 0$, the boundary values of the function $\varphi'(z)$ may admit at this point a singularity of order less than unity.

Along with the lower half-plane which we now denote by S^-, let us introduce into consideration the upper half-plane S^+ and, following N. I. Muskhelishvili ([93], Sec. 112, formula (16)), extend the definition of the function $\varphi(z)$ to S^+, assuming

$$\varphi(z) = -z\overline{\varphi}'(z) - \overline{\psi}(z) \qquad (z \text{ in } S^+). \tag{30.11}$$

On the basis of the boundary condition (30.6) the function (30.11), holomorphic in S^+, analytically continues the values of the complex potential $\varphi(z)$ from S^- through the semi-axis $(-\infty, 0)$. This extended function, which we again denote by $\varphi(z)$, is sectionally holomorphic in the region S representing the entire z plane cut along the semi-axis $(0, \infty)$. The function $\psi(z)$ can be expressed in terms of $\varphi(z)$ from formula (30.11) as follows:

$$\psi(z) = -\overline{\varphi}(z) - z\varphi'(z) \qquad (z \text{ in } S^-). \tag{30.12}$$

Formulas (30.11) and (30.12) suggest) that representation (30.9) is equally valid for the sectionally holomorphic function $\varphi(z)$.

Considering the plane cut along the positive part of the real axis, we shall distinguish between the upper and lower sides of the cut and assign to the quantities pertaining to these sides a plus and a minus sign, respectively.

We replace the functions $\psi(t)$, $\psi'(t)$ [or rather, $\psi^-(t)$, $\psi'^-(t)$] in the boundary conditions (30.6), (30.7) by their values (30.12). These conditions then become

$$\varphi^-(t) - \varphi^+(t) = 0 \qquad (t < 0). \tag{30.13'}$$

$$ip + h[\varphi^-(t) - \varphi^+(t)] + ik\,\mathrm{Re}\,[\varphi'^+(t) + 3\varphi'^-(t)] = 0 \qquad (t > 0). \tag{30.13''}$$

To solve this problem we assume in the region S

$$\varphi(z) = -\frac{X}{2\pi}\ln z + \varphi_0(z), \tag{30.14}$$

$$\varphi_0(z) = \frac{1}{2\pi i}\int_0^\infty \frac{\omega(\tau)\,d\tau}{\tau - z}, \tag{30.15}$$

where $\omega(t) = \mu(t) + i\nu(t)$ is a new unknown function on $[0, \infty)$ and by $\ln z$ is meant that branch of this function which takes, say, the real values on the upper side of the cut.

As to the function $\omega(t)$, we assume that, together with its first-order derivative, it satisfies the Hölder condition on any finite interval, not having $t = 0$ for its end, and that $\omega(t)$ belongs to the class $L_p\,[0, \infty)$ with a certain $p > 1$, and $\omega'(t)$ to the class $L\,[0, \infty)$. We have

$$\varphi_0'(z) = -\frac{\omega(0)}{2\pi i z} + \frac{1}{2\pi i}\int_0^\infty \frac{\omega'(\tau)\,d\tau}{\tau - z}, \tag{30.16}$$

$$\varphi_0^\pm(t) = \pm\frac{1}{2}\omega(t) + \frac{1}{2\pi i}\int_0^\infty \frac{\omega(\tau)\,d\tau}{\tau - t}, \tag{30.17}$$

$$\varphi_0'^\pm(t) = -\frac{\omega(0)}{2\pi i t} \pm \frac{1}{2}\omega'(t) + \frac{1}{2\pi i}\int_0^\infty \frac{\omega'(\tau)\,d\tau}{\tau - t}. \tag{30.18}$$

Inserting the above expressions for the boundary values of the function $\varphi(z)$ and its derivative in formulas (30.13) and taking into account that

$$\varphi^-(t)-\varphi^+(t)=-\omega(t)-\frac{X}{2\pi}[\ln^- t-\ln^+ t]=$$
$$=-\omega(t)-iX=-\omega(t)-\frac{ip_0}{h},$$

we obtain

$$h\omega(t)+ik\left\{\frac{2[X+\nu(0)]}{\pi t}+\mu'(t)-\frac{2}{\pi}\int_0^\infty\frac{\nu'(\tau)\,d\tau}{\tau-t}\right\}=0. \qquad (30.19)$$

Let us try to determine the quantity $\nu(0)$. According to formulas (30.4) we have

$$\tau_{xy}^-=\operatorname{Im}\{t\varphi''^-(t)+\psi'^-(t)\}.$$

If the function $\psi'^-(t)$ is replaced here by its expression from formula (30.12), the right-hand side of the equality transforms to

$$\operatorname{Im}\{t\varphi''^-(t)+\psi'^-(t)\}=-\operatorname{Im}\{\overline{\varphi'^+(t)}+\varphi'^-(t)\}.$$

For the limiting values of the function $\varphi'(z)$ in the above equality we make use of the Sokhotskii-Plemelj formulas (30.18). We then obtain an important relation given by

$$\tau_{xy}^-=\nu'(t) \qquad (t>0). \qquad (30.20)$$

Let us now turn our attention to the first equality of (30.2). By virtue of relation (30.20) it takes the form

$$p_0-h[\nu(x)-\nu(0)]+k\sigma_x=0 \qquad (x>0).$$

The limiting process when $x\to\infty$ determines $\nu(0)$ as follows:

$$\nu(0)=-\frac{p_0}{h}. \qquad (30.21)$$

It will now be noted that, by virtue of formulas (30.8) and (30.21), the first term in the braces of formula (30.19) drops out and Eq. (30.19), after separating the real and imaginary parts in it, is represented in the form of the following two equations:

$$\mu(t)=0,$$
$$\nu(t)-\frac{\lambda}{\pi}\int_0^\infty\frac{\nu'(\tau)\,d\tau}{\tau-t}=0 \qquad (t>0), \qquad (30.22)$$

where

$$\lambda=\frac{2k}{h}=\frac{2E_0S_0}{Eh}. \qquad (30.23)$$

Thus, for the determination of the density function of integral (30.15) we have obtained the homogeneous singular integro-differential equation (30.22) subject to the additional condition (30.21).

Note. Equation (30.22) can easily be given the form of a nonhomogeneous singular integral equation. Indeed, from equalities (30.20) and (30.21) it follows that

$$\nu(t) = \int_0^t \tau_{xy}^- \, d\xi - \frac{p_0}{h}.$$

inserting this expression together with (30.20) in Eq. (30.22), we find

$$\frac{\lambda}{\pi} \int_0^\infty \frac{\tau_{xy}^- \, d\xi}{\xi - t} - \int_0^t \tau_{xy}^- \, d\xi + \frac{p_0}{h} = 0. \tag{30.24}$$

Equation (30.24) has been obtained from other considerations in the above-cited work of W. T. Koiter [59].

31. Infinite Plane with a Semi-infinite Stringer

Let an infinite plate occupying the entire plane of the complex variable $z = x + iy$ be reinforced with a solid stringer continuously attached to it along the semi-axis $x > 0$. Suppose that the tip of the stringer at the point $z = 0$ is acted on by the same axial force p_0 and, in addition, the medium is subjected at infinity to biaxial tension with intensities P and Q, respectively, in the direction of the x and y axes.

Furthermore, we introduce into consideration the normal tensile force N at the section of the bar, the normal stress σ_x^0 in the bar and the extension ε^0 of the axis of the bar. We shall distinguish between the left and right (in relation to the positive x direction) sides of the bar axis and arbitrarily assign to the quantities pertaining to these sides a plus and a minus sign, respectively.

Consider a part of the stringer enclosed between two sections with co-ordinates x and $x + dx$ and calculate the components of the resultant of the external forces applied to its surface. The face of the isolated element on the side of y increasing is acted on in the x direction by the force $h\tau_{xy}^+ \, dx$ and the opposite face by the force $-h\tau_{xy}^- \, dx$. The bases of the element perpendicular to the x axis are acted on by forces equal to $-N(x)$ and $N(x + dx)$. Consequently, the projection on the x axis of all forces acting on the isolated part of the stringer is

$$h\,[\tau_{xy}^+ - \tau_{xy}^-]\, dx + N(x + dx) - N(x)$$

and the equilibrium condition gives

$$h\,[\tau_{xy}^+ - \tau_{xy}^-]\, dx + N(x + dx) - N(x) = 0 \qquad (x > 0). \tag{31.1}$$

According to our assumption that the stringer has no flexural rigidity, there will be no bending moment at its sections, and hence

$$\sigma_y^+ - \sigma_y^- = 0 \qquad (x > 0). \tag{31.1a}$$

If the above two equalities are taken together and if, further, use is made of the fact that

$$N(x) = -p_0 \qquad (x = 0),$$

the condition for the equilibrium of any finite part $(0, x)$ of the stringer is obtained in the form

$$h \int_0^x [(\tau_{xy}^+ + i\sigma_y^+) - (\tau_{xy}^- + i\sigma_y^-)]\, dt + p_0 + N(x) = 0. \tag{31.2}$$

This equation is identical with Eq. (30.3) if we assume in the former $\tau_{xy}^+ = \sigma_y^+ = 0$ for all $x > 0$ and also take into account that $N(x) = k\sigma_x$.*

To obtain a complete set of boundary conditions of our problem, it is necessary, in addition to condition (31.2), to express conditions for continuity of the displacement components and the extension $\varepsilon_x = \frac{\partial u}{\partial x}$ across the axis of the stringer. The continuity conditions are of the form

$$u^+ + iv^+ = u^- + iv^-, \qquad \varepsilon_x^+ = \varepsilon_x^- = \varepsilon^0 \qquad (x > 0). \tag{31.3}$$

The above condition of equality of extensions, together with the formula

$$N = E_0 S_0 \varepsilon^0$$

which is well known from the theory of small deformations of curved bars, allows Eq. (31.2) to be written as

$$h \int_0^x [(\tau_{xy}^+ + i\sigma_y^+) - (\tau_{xy}^- + i\sigma_y^-)]\, dt + p_0 + E_0 S_0 \left(\frac{\partial u}{\partial x}\right)^+ = 0 \qquad (x > 0). \tag{31.4}$$

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* The validity of the equality can easily be verified on the basis of the formula for $N(x)$ given in the text following (31.3). Indeed, by virtue of the equality of the extensions for the stringer axis and the boundary of the half-plane under the stringer,

$$N(x) = E_0 S_0 \varepsilon^0(x) = E_0 S_0 \frac{\partial u}{\partial x} \qquad (x > 0).$$

On the other hand, according to Hooke's law for a state of generalized plane stress,

$$\frac{\partial u}{\partial x} = \frac{1}{E}(\sigma_x - \nu\sigma_y).$$

It remains to note that under the stringer $\sigma_y = 0$, and, consequently,

$$N(x) = k\sigma_x, \quad k = \frac{E_0 S_0}{E} \qquad (x > 0).$$

Relation (31.4), together with the first group of conditions (31.3), determines the boundary conditions of our problem.

By using, as before, formulas (30.4), (30.5) and also the complex representation formula for displacements (1.3)

$$2\mu\,(u + iv) = \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}, \tag{31.5}$$

we can represent the boundary conditions of the problem in the form

$$\Omega^-(t, \bar{t}) - \Omega^+(t, \bar{t}) + \\ + iK_0 \operatorname{Re}\left[\varkappa\varphi'^+(t) - \overline{\varphi'^+(t)} - t\overline{\varphi''^+(t)} - \overline{\psi'^+(t)}\right] + \frac{ip_0}{h} = 0, \tag{31.6}$$

$$\varkappa\varphi^+(t) - t\overline{\varphi'^+(t)} - \overline{\psi^+(t)} = \varkappa\varphi^-(t) - t\overline{\varphi'^-(t)} - \overline{\psi^-(t)} \qquad (t > 0), \tag{31.7}$$

where the following notation has been introduced:

$$\Omega(z, \bar{z}) = \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}, \qquad K_0 = \frac{E_0 S_0}{2\mu h}. \tag{31.8}$$

Thus, we come to the problem of determining the analytic functions $\varphi(z)$, $\psi(z)$ in the region S, representing the entire z plane cut along the semi-axis L ($x > 0$, $y = 0$), from the boundary conditions (31.6) and (31.7).

The functions $\varphi(z)$ and $\psi(z)$ are, of course, sectionally holomorphic in the region S and, according to our condition imposed on the stress field in infinitely remote parts of the plane, for large $|z|$ are of the form (cf. N. I. Muskhelishvili [93], A. I. Kalandiya [44])

$$\left.\begin{aligned} \varphi(z) &= -A \ln z + \Gamma z + \varphi_0(z), \\ \psi(z) &= \varkappa A \ln z + \Gamma' z + \psi_0(z), \end{aligned}\right\} \tag{31.9}$$

where A is a real constant expressible in terms of the components of the resultant of the external forces applied to the cut L

$$\Gamma = \frac{Q+P}{4}, \qquad \Gamma' = \frac{Q-P}{2} \tag{31.10}$$

and φ_0, ψ_0 are also sectionally holomorphic functions in S which admit the asymptotic representation

$$\varphi_0(z) = o(1) + \text{constant}, \quad \psi_0(z) = o(1) + \text{constant}. \tag{31.11}$$

Assume in the region S

$$\chi(z) = z\varphi'(z) + \psi(z) \tag{31.12}$$

and write condition (31.7) as

$$\varkappa\varphi^+(t) + \overline{\chi^-(t)} = \varkappa\varphi^-(t) + \overline{\chi^+(t)} \text{ on } L,$$

or, what is equivalent,

$$\varkappa\varphi^+(t) + \bar{\chi}^+(t) = \varkappa\varphi^-(t) + \bar{\chi}^-(t) \text{ on } L. \tag{31.13}$$

On the basis of formulas (31.9) to (31.11) the function $\varkappa\varphi(z) + \overline{\chi}(z)$ for large $|z|$ is of the form

$$\varkappa\varphi(z)+\overline{\chi}(z)=\left[\frac{\varkappa+3}{4}Q+\frac{\varkappa-1}{4}P\right]z+o(1)+\text{constant}$$

and, according to condition (31.13), can be continued analytically through L. Hence, throughout the z plane

$$\varkappa\varphi(z)+\overline{\chi}(z)=Bz+\text{constant}, \tag{31.14}$$

where

$$B=\frac{\varkappa+3}{4}Q+\frac{\varkappa-1}{4}P. \tag{31.15}$$

From formula (31.14) we find

$$\left.\begin{aligned}\overline{\chi^{+}(t)}&=-\varkappa\varphi^{-}(t)+Bt+\text{constant},\\ \overline{\chi^{-}(t)}&=-\varkappa\varphi^{+}(t)+Bt+\text{constant}.\end{aligned}\right\} \tag{31.16}$$

Further we have

$$\left.\begin{aligned}t\overline{\varphi''^{+}(t)}+\overline{\psi'^{+}(t)}&=\overline{\chi'^{+}(t)}-\overline{\varphi'^{+}(t)}=-\varkappa\varphi'^{-}(t)-\overline{\varphi'^{+}(t)}+B,\\ t\overline{\varphi''^{-}(t)}+\overline{\psi'^{-}(t)}&=\overline{\chi'^{-}(t)}-\overline{\varphi'^{-}(t)}=-\varkappa\varphi'^{+}(t)-\overline{\varphi'^{-}(t)}+B.\end{aligned}\right\} \tag{31.17}$$

Relations (31.16) and (31.17) enable us to eliminate the function ψ from the boundary condition (31.6) and to reduce our problem to the determination of the single function $\varphi(z)$, sectionally holomorphic in the region S, from the boundary condition

$$\varphi^{-}(t)-\varphi^{+}(t)+i\lambda\,\mathrm{Re}\,[\varphi'^{+}(t)+\varphi'^{-}(t)]+\frac{i}{\varkappa+1}\left[\frac{p_0}{h}-K_0\bar{B}\right]=0 \text{ on } L, \tag{31.18}$$

where

$$\lambda=\frac{E_0S_0\varkappa}{2\mu h(\varkappa+1)}. \tag{31.19}$$

The solution of problem (31.18) is sought in the form

$$\varphi(z)=-\frac{X}{2\pi(1+x)}\ln z+\frac{P+Q}{4}z+\varphi_0(z), \tag{31.20}$$

$$\varphi_0(z)=\frac{1}{2\pi i}\int\limits_0^\infty\frac{\omega(\tau)\,d\tau}{\tau-z}, \tag{31.21}$$

where X denotes the scalar value of the resultant of the external forces applied to the cut L

$$X=-\int\limits_0^\infty(\tau_{xy}^{+}-\tau_{xy}^{-})\,dt \tag{31.22}$$

and $\omega(t)$ is a new unknown function on L subject to the above-mentioned conditions.

The function $\varphi_0(z)$ is obviously sectionally holomorphic in the cut plane and is bounded for a large $|z|$. So far as its behaviour near $z = 0$ is concerned, it is determined by the condition of the physical problem which admits a singularity of order less than unity for the derivative $\varphi'(z)$ in the neighbourhood considered. By virtue of representation (31.20) this means that the function $\varphi_0(z)$ itself must have a singularity of the type $\ln z$ near $z = 0$.

If the boundary values of the function $\varphi(z)$ and its derivative defined by the Sokhotskii-Plemelj formulas are substituted in condition (31.18), after elementary computations we find an equation for $\omega(t)$:

$$-\omega(t) + i\lambda\left\{-\frac{1}{\pi t}\left[\frac{X}{\varkappa+1} + \nu(0)\right] + \frac{1}{\pi}\int_0^\infty \frac{\nu'(\tau)\,d\tau}{\tau - t}\right\} +$$

$$+\frac{i}{(\varkappa+1)}\left[\frac{p_0}{h} - X + K_0\Omega_0\right] = 0 \quad \text{on } L, \qquad (31.23)$$

where the constants λ and K_0 are defined by equalities (31.19) and (31.8), and

$$\Omega_0 = \frac{\varkappa+1}{4}P + \frac{\varkappa-3}{4}Q. \qquad (31.24)$$

By formula (31.5) we calculate the extension ε_x in the uniform field corresponding to the given tensile forces P and Q. By using the potentials

$$\varphi(z) = \frac{Q+P}{4}z, \qquad \psi(z) = \frac{Q-P}{2}z,$$

we easily find

$$\frac{\partial u}{\partial x} = \frac{1}{2\mu}\Omega_0.$$

From this, for the normal force $N(x)$ at the sections of the stringer we have the limiting equality

$$N(x) = E_0S_0\left(\frac{\partial u}{\partial x}\right)^+ = \frac{E_0S_0}{2\mu}\Omega_0 \qquad (\text{as } x \to \infty). \qquad (31.25)$$

In the real part of equality (31.4) we pass to the limit as $x \to \infty$ and take into account formulas (31.22) and (31.25). We then obtain a relation which determines the magnitude of the resultant of the forces exerted by the plate on the stringer

$$X = \frac{p_0}{h} + K_0\Omega_0. \qquad (31.26)$$

On the basis of relations (31.17) we further have

$$-\tau_{xy}^- = \operatorname{Im}\{t\overline{\varphi''^-(t)} + \overline{\psi'^-(t)}\} = -\operatorname{Im}\{\varkappa\varphi'^+(t) + \overline{\varphi'^-(t)} + B\},$$

$$-\tau_{xy}^+ = \operatorname{Im}\{t\overline{\varphi''^+(t)} + \overline{\psi'^+(t)}\} = -\operatorname{Im}\{\varkappa\varphi'^-(t) + \overline{\varphi'^+(t)} + B\} \text{ on } L.$$

From this we obtain an equality

$$\tau_{xy}^{+}-\tau_{xy}^{-}=-(\varkappa+1)\operatorname{Im}\{\varphi'^{+}(t)-\varphi'^{-}(t)\}$$

which, by virtue of representation (31.20) and the Sokhotskii-Plemelj formulas (30.18), immediately gives

$$\tau_{xy}^{+}-\tau_{xy}^{-}=-(\varkappa+1)\,\nu'(t)\quad\text{on } L. \tag{31.27}$$

By using the above relation, we represent the real part of equality (31.4) in the form

$$-h(\varkappa+1)[\nu(x)-\nu(0)]+p_0+E_0S_0\left(\frac{\partial u}{\partial x}\right)^{+}=0 \qquad (x>0).$$

Passing here to the limit as $x\to\infty$ and taking into account relation (31.25), we find

$$\nu(0)=-\frac{1}{\varkappa+1}\left[\frac{p_0}{h}+K_0\Omega_0\right]. \tag{31.28}$$

On the basis of formulas (31.26) and (31.28) the coefficient of $\frac{1}{t}$ in Eq. (31.23) and the last bracketed term vanish and the equation falls into a pair of real equations of the form

$$\mu(t)=0 \text{ on } L, \qquad \nu(t)-\frac{\lambda}{\pi}\int_0^{\infty}\frac{\nu'(\tau)\,d\tau}{\tau-t}=0 \text{ on } L, \tag{31.29}$$

where λ is defined by formula (31.19) and the unknown function $\nu(t)$ is subject to the additional condition (31.28).

Equation (31.29) is precisely the integro-differential equation of our problem.

In the special case when the elastic medium is not subjected to external forces at infinity ($P=Q=0$) we obviously have

$$\tau_{xy}^{-}=-\tau_{xy}^{+}, \qquad 2\tau_{xy}^{-}=(\varkappa+1)\,\nu'(t), \qquad \nu(0)=-\frac{p_0}{(\varkappa+1)\,h}.$$

Substituting in Eq. (31.29) the formula

$$\nu(t)=\frac{2}{(\varkappa+1)}\int_0^{t}\tau_{xy}^{-}\,dx-\frac{p_0}{h(\varkappa+1)},$$

we then obtain an equation

$$\frac{\lambda}{\pi}\int_0^{\infty}\frac{\tau_{xy}^{-}\,d\xi}{\xi-t}-\int_0^{t}\tau_{xy}^{-}\,d\xi+\frac{p_0}{2h}=0 \text{ on } L \tag{31.30}$$

which has been found in a different way in the above-cited work of W. T. Koiter [59].*

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* See the note at the end of the preceding section.

Finally, it is of interest to note that in the case when the external forces p_0, P, Q are interrelated by the equation

$$p_0+\frac{E_0S_0}{2\mu}\left[\frac{\varkappa+1}{4}P+\frac{\varkappa-3}{4}Q\right]=0$$

we have

$$X=0, \qquad \varphi_0(z)=0 \qquad (\text{in } S),$$

and consequently the resulting stress field in the plate is everywhere uniform. Noting that

$$\varkappa=\frac{3-\nu}{1+\nu},$$

where ν is Poisson's ratio, the condition for the uniformity of the field may be written in the following simpler form:

$$p_0+\frac{E_0S_0}{E}[P-\nu Q]=0. \tag{31.31}$$

32. Solution of the Integro-differential Equation

The integro-differential equations of the problems derived in the preceding two sections are exactly similar in form and are given by

$$\left.\begin{aligned} &\nu(t)-\frac{\lambda}{\pi}\int\limits_0^\infty\frac{\nu'(\tau)\,d\tau}{\tau-t}=0 \text{ on } L,\\ &\nu(0)=a, \end{aligned}\right\} \tag{32.1}$$

where λ is a positive parameter and a is a given number.

Let us try to apply the Wiener-Hopf method to the solution of (32.1). Along with Eqs. (32.1) we introduce into consideration the following integro-differential equation:

$$\left.\begin{aligned} &g(t)-\frac{\lambda}{\pi}\int\limits_{-\infty}^{+\infty}\frac{g'(\tau)\,d\tau}{\tau-t}=b(t) \qquad (-\infty<t<\infty),\\ &g(0)=a, \end{aligned}\right\} \tag{32.2}$$

where

$$b(t)=0 \qquad (t\geqslant 0), \qquad b(t)=-\frac{\lambda}{\pi}\int\limits_{-\infty}^{\infty}\frac{g'(\tau)\,d\tau}{\tau-t} \qquad (t<0). \tag{32.3}$$

If the derivative of the function

$$g(t)=\begin{cases}\nu(t), & t\geqslant 0,\\ 0, & t<0,\end{cases} \tag{32.4}$$

where $\nu(t)$ is the solution of Eq. (32.1) with the required properties, were summable on the axis, Eqs. (32.1) and (32.2) would be equivalent to each other. However, the derivative $g'(t)$, as appears from definition (32.4), represents the sum of an integrable function and a function of the type $\delta(t)$. In order to obtain the desired solution $\nu(t)$ of Eq. (32.1) from (32.2) we must, therefore, reject the assumption that the derivative of the solution of (32.2) is summable and perform all necessary operations on this solution in a purely formal way.

Below we shall use the techniques of Fourier integrals. By convention we designate functions by low-case letters, and their Fourier transforms by the respective upper-case letters.

On the basis of formulas (32.2) and (32.3) we have

$$\int_{-\infty}^{\infty} g'(\tau)\, e^{it\tau}\, d\tau = g(\tau)\, e^{it\tau}\Big|_0^{\infty} - it\int_0^{\infty} g(\tau)\, e^{it\tau}\, d\tau = -a - itG(t). \qquad (32.5)$$

Moreover,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} e^{ixt}\frac{dx}{x-\tau} = ie^{it\tau}\operatorname{sgn} t. \qquad (32.6)$$

Performing the Fourier transformation on both sides of the first of equalities (32.2) and taking into account relations (32.5) and (32.6), we have

$$G(t) - i\lambda\,[a + itG(t)]\operatorname{sgn} t = B(t) \qquad (-\infty < t < \infty).$$

The functions $G(t)$ and $B(t)$ represent, by definition, the limiting values of the functions $G(z)$ and $B(z)$ holomorphic in S^+ and S^-, respectively. The preceding equality may therefore be written as

$$G^+(t)\,[1 + \lambda\,|t|] = B^-(t) + i\lambda a\operatorname{sgn} t. \qquad (32.7)$$

Thus, for the determination of the unknown Fourier transform $G(t)$ of the solution of Eq. (32.2) we have obtained the problem of linear relationship (32.7) on the axis.

We represent the coefficient of $G^+(t)$ in formula (32.7) as

$$1+\lambda\,|t| = \frac{1+\lambda\,|t|}{\sqrt{1+\lambda^2t^2}}\sqrt{1+i\lambda t}\,\sqrt{1-i\lambda t}$$

and introduce into consideration a canonical solution $X(z)$ of the auxiliary problem of linear relationship

$$X^+(t) = \frac{1+\lambda\,|t|}{\sqrt{1+\lambda^2t^2}}\,X^-(t). \qquad (32.8)$$

For $X(z)$ let us take the function

$$X(z) = \exp\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty}\ln\frac{1+\lambda\,|\tau|}{\sqrt{1+\lambda^2\tau^2}}\,\frac{d\tau}{\tau-z}\right]. \qquad (32.9)$$

It is easily verified that $X(z)$ satisfies the boundary condition (32.8), nowhere vanishes, including the real axis, and as $|z| \to \infty$ in the closed half-planes it has the values

$$X^{\pm}(\infty) = 1. \qquad (32.10)$$

We now represent the boundary condition (32.7) in the form

$$G^{+}(t)\sqrt{1-i\lambda t}\,X^{+}(t) = \frac{B^{-}(t)\,X^{-}(t)}{\sqrt{1+i\lambda t}} + \frac{i\lambda a X^{-}(t)\operatorname{sgn} t}{\sqrt{1+i\lambda t}}. \qquad (32.11)$$

From this it follows immediately that

$$\sqrt{1-i\lambda z}\,X(z)\,G(z) = \frac{\lambda a}{2\pi}\int_{-\infty}^{\infty} \frac{X^{-}(t)\operatorname{sgn} t}{\sqrt{1+i\lambda t}}\,\frac{dt}{t-z} \qquad (z \text{ in } S^{+})$$

or, by the well-known Cauchy theorem,

$$\sqrt{1-i\lambda z}\,X(z)\,G(z) = \frac{\lambda a}{\pi}\int_{0}^{\infty} \frac{X^{-}(t)}{\sqrt{1+i\lambda t}}\,\frac{dt}{t-z} \qquad (z \text{ in } S^{+}). \quad (32.12)$$

Let us now determine the Fourier transform of the function $g'(t)$. Designate it as $G_1(t)$ and note that the values of this function represent the boundary values $G_1^{+}(t)$ of a function holomorphic in z in the half-plane S^{+}. Taking this into account, we rewrite equality (32.5) in the alternative form

$$G_1(z) = -a - izG(z). \qquad (32.13)$$

Substituting $G(z)$ from relation (32.12), we find

$$G_1(z) = -a + \frac{\lambda a z}{\sqrt{1-i\lambda z}\,X(z)}\,\frac{1}{\pi i}\int_{0}^{\infty} \frac{X^{-}(t)}{\sqrt{1+i\lambda t}}\,\frac{dt}{t-z}. \qquad (32.14)$$

Calculate the limit of the function $G_1(z)$ when the point z goes into infinity remaining all the time in the upper half-plane. To facilitate the calculation, we make a change of the variable z and also a change of the integration variable in expression (32.14) by substitutions

$$z = -\frac{1}{\zeta}, \qquad t = -\frac{1}{\tau}. \qquad (32.15)$$

Transformation (32.15) maps the upper half-plane of the z plane into the upper half-plane of the ζ plane, the real axis into the real one and the neighbourhood of the point at infinity into the neighbourhood of the point $\zeta = 0$. Equality (32.14) then transforms to

$$G_1^{*}(\zeta) + a = -\frac{\sqrt{\zeta}}{\sqrt{\zeta+i\lambda}\,X^{*}(\zeta)}\,\frac{\lambda a}{\pi i}\int_{0}^{\infty} \frac{X^{*-}(\tau)}{\sqrt{\tau(\tau-i\lambda)}}\,\frac{d\tau}{\tau-\zeta} \qquad (32.16)$$

with $G_1^{*}(\zeta) = G_1(z)$, $X^{*}(\zeta) = X(z)$.

To estimate equality (32.16) for small $|\zeta|$ we make use of Muskhelishvili's formula (3.3) and the limiting equality (32.10). The following formula holds for points ζ lying in the neighbourhood of the origin in the cut plane S:

$$G_1^*(\zeta) + a = -a + o(1). \tag{32.17}$$

From this, returning to formula (32.14), we conclude that the holomorphic function $G_0(z)$ in S^+ defined by the equality

$$G_0(z) = G_1(z) + 2a = a + \frac{\lambda a z}{\sqrt{1 - i\lambda z}\, X(z)} \frac{1}{\pi i} \int_0^\infty \frac{X^-(t)}{\sqrt{1 + i\lambda t}} \frac{dt}{t - z} \tag{32.18}$$

vanishes at infinity. Its boundary values $G_0^+(t)$ represent the Fourier transform of the unknown function $\nu'(t)$:

$$\nu'(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_0^+(\tau)\, e^{-it\tau}\, d\tau. \tag{32.19}$$

The above equality completely determines the required solution $\nu(t)$ of Eq. (32.1).

Consequently, the solutions of the boundary value problems considered in the preceding two sections are given by formulas (30.14) and (31.20), where $\varphi_0(z)$, a sectionally holomorphic function in the region S, represents a Cauchy-type integral over a half-line whose density function is determined by the inverse Fourier transform (32.19).

The problem of a semi-infinite stringer is thus solved in closed form.*

33. Stringer of Finite Length

In the conditions of Sec. 30 it will be assumed that the stringer is of finite length and is situated over the segment $[-a, a]$ of the real axis. As in Sec. 30, to the left end of the stringer is applied an axial load p_0 and the half-plane is extended at infinity by forces parallel to the boundary of the medium (Fig. 4).

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* The solution of the integro-differential equation (32.1), closed to the same degree as (32.19), was first indicated in the work of W. T. Koiter [59]. Starting out from Eq. (31.30) and applying the Mellin transformation to it, the above-named author arrives at a finite-difference equation in a strip of the plane of an auxiliary complex argument. This equation is solved explicitly using the Laplace transformation and then the solution of the original equation (31.30) is determined by the inverse Mellin transformation. It should be recognized that the final formulas for the shearing stresses τ_{xy} and the stresses in the stringer $N(x)$ derived by W. T. Koiter are elegant in form, compact and easy to treat by numerical analysis.

The problem is by no means new. It appears to have been first considered by A. Pflüger [101], and was afterwards studied by many authors. Among recent works devoted to its solution we may mention the work of N. Kh. Arutyunyan [4], G. A. Morar' and G. Ya. Popov [84], F. Erdogan and G. D. Gupta [21].

Below is given a method for the solution of this problem which is general in a sense and at the same time rather efficient. Its generality consists essentially in providing a means of describing the stress field universally, that is, in the entire half-plane, including the boundary.*

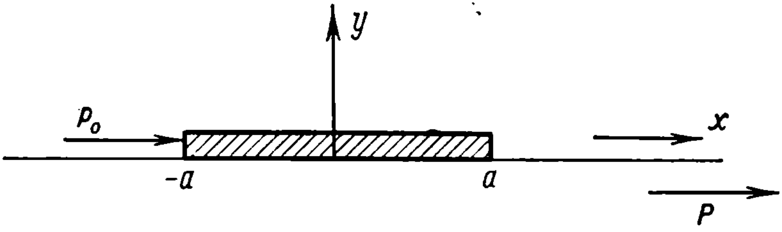

Fig. 4

Proceeding to the solution, we note that the boundary of the half-plane outside the interval $|x| \leqslant a$ is not loaded and, by hypothesis, the stringer offers no resistance to bending. The boundary conditions of the problem are therefore written as

$$p_0 - h\int_{-a}^{x} \tau_{xy}\,dt + k\sigma_x = 0, \quad \sigma_y = 0 \qquad (|x| \leqslant a), \tag{33.1}$$

$$\sigma_y = \tau_{xy} = 0 \qquad (|x| > a). \tag{33.2}$$

Equalities (33.1) express the condition of equilibrium of any part (—a, x) of the stringer bonded to the half-plane provided that the extensions of the stringer axis and of the adjacent boundary of the half-plane are equal.

It is supposed, for the present, that the tangential forces τ_{xy} under the stringer are given throughout the interval (—a, a). Problem (33.1), (33.2) is then reduced to the common problem of an elastic half-plane with given external forces and its solution is represented by the well-known formulas (N. I. Muskhelishvili [93],

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* In the last two of the above-mentioned works use is made of a somewhat different efficient method of solution. The idea of the method is the representation of the required solution of the singular equation of the problem as series in Chebyshev's polynomials and the subsequent reduction of the solution to an infinite system of linear algebraic equations. As far as we know, this method was first suggested by G. Ya. Popov [104].

Sec. 93; see also A. I. Kalandiya [44])

$$\left.\begin{aligned}\Phi(z)&=\frac{P}{4}+\frac{1}{2\pi}\int_L\frac{\tau_{xy}\,dt}{t-z},\\ -\Psi(z)&=\frac{P}{4}+\frac{1}{2\pi}\int_L\frac{\tau_{xy}\,dt}{t-z}+\Phi(z)+z\Phi'(z),\end{aligned}\right\}\qquad(33.3)$$

$$\Phi(z)=\varphi'(z),\qquad \Psi(z)=\psi'(z),\qquad z=x+iy,\qquad y<0.$$

Consequently, when the forces τ_{xy} under the stringer have been determined, formulas (33.3) give, according to the Kolosov-Muskhelishvili representations (1.2), the stress field in the entire half-plane.*

For the stress components we have

$$\left.\begin{aligned}\sigma_x&=\operatorname{Re}\left\{\frac{2}{\pi}\int_L\frac{\tau_{xy}\,dt}{t-z}+(z-\bar{z})\,\Phi'(z)\right\}+P,\\ \sigma_y&=-\operatorname{Re}\{(z-\bar{z})\,\Phi'(z)\},\\ \tau_{xy}&=-\operatorname{Im}\left\{\frac{1}{\pi}\int_L\frac{\tau_{xy}\,dt}{t-z}+(z-\bar{z})\,\Phi'(z)\right\}.\end{aligned}\right\}\qquad(33.4)$$

As usual, it is assumed that for any point t of the x axis, with the possible exception of the points $t=\pm a$,

$$\lim_{y\to 0} y\Phi'(z)=\lim_{y\to 0} y\Phi'(t+iy)=0.$$

From the first formula of (33.4), by allowing the point z to approach an arbitrary point x of the real axis not coinciding with the ends of the interval, we find

$$\sigma_x=\frac{2}{\pi}\int_L\frac{\tau_{xy}\,dt}{t-x}+P\qquad(-\infty<x<\infty,\ x\neq\pm a).\qquad(33.5)$$

For points x from the interval $(-a,\ a)$ the integral is to be understood as a Cauchy principal value.

To obtain a relation for the unknown forces τ_{xy} under the stringer, it is now sufficient to insert σ_x from (33.5) in the first of equalities (33.1):

$$p_0-h\int_{-a}^{x}\tau_{xy}\,dt+k\sigma_x=0\qquad(|x|\leqslant a).\qquad(33.6)$$

Thus,

$$\frac{\lambda}{\pi}\int_{-a}^{a}\frac{\tau_{xy}\,dt}{t-x}-\int_{-a}^{x}\tau_{xy}\,dt+\frac{p_0}{h}+\frac{\lambda}{2}P=0\qquad(|x|<a),\qquad(33.7)$$

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* The stresses in the stringer $\sigma_x^0=\frac{E_0}{E}\sigma_x$ are determined directly from the first of equalities (33.1).

where λ has the value

$$\lambda = \frac{2E_0S_0}{Eh}. \tag{33.8}$$

No external forces are applied to the right end of the stringer in our example. Equality (33.6) with $x = a$ therefore gives the condition of equilibrium of the stringer as

$$\int_{-a}^{a} \tau_{xy}\, dt = \frac{p_0}{h}. \tag{33.9}$$

The same equality with $x = -a$ leads to the relation

$$p_0 = -kQ, \tag{33.10}$$

where $\frac{E_0}{E} Q$ denotes the intensity of forces σ_x^0 acting on the left end.

The set of equalities (33.7) and (33.9) determines the basic singular equation of the problem, an equation that is well known in the literature. With (33.8) and with the notation $\tau_{xy}(x, 0) = \tau(x)$ it takes the form

$$\left.\begin{aligned} &\frac{1}{\pi}\int_{-a}^{a} \frac{\tau(t)\, dt}{t-x} - \frac{1}{\pi}\int_{-a}^{a} H(x-t)\,\tau(t)\, dt = \frac{1}{2}(Q-P), \\ &\int_{-a}^{a} \tau(t)\, dt = -\frac{\lambda}{2} Q \qquad (|x| < a), \end{aligned}\right\} \tag{33.11}$$

where H is the Heaviside unit function.

When the function $\tau(x)$ has been found from (33.11), we can determine all unknowns of the problem. In fracture mechanics the quantities of greatest interest are Irwin's stress intensity factors defined by the formulas

$$\left.\begin{aligned} T_{-a} &= \lim_{x \to -a} \sqrt{2(a+x)}\,\tau(x), \\ T_a &= \lim_{x \to a} \sqrt{2(a-x)}\,\tau(x). \end{aligned}\right\} \tag{33.12}$$

Introducing non-dimensional quantities ξ, η by a change $x = a\xi$, $t = a\eta$, we reduce system (33.11) to the standard form

$$\left.\begin{aligned} &\frac{1}{\pi}\int_{-1}^{1} \frac{\tau(\eta)\, d\eta}{\eta-\xi} - \alpha\int_{-1}^{1} H(\xi-\eta)\,\tau(\eta)\, d\eta = \frac{1}{2}(Q-P), \\ &\int_{-1}^{1} \tau(\eta)\, d\eta = -\frac{1}{2\alpha} Q \qquad (|\xi| < 1). \end{aligned}\right\} \tag{33.13}$$

Here use has been made of the notation

$$\tau(\xi)=\tau(x), \qquad \alpha=\frac{a}{\lambda},$$

where λ is defined by formula (33.8).

For an approximate solution of Eq. (33.13) we employ the method of Sec. 13.* As is known, the solution is not bounded at either of the ends of the interval and is representable in the form

$$\tau(\xi)=\frac{\tau_0(\xi)}{\sqrt{1-\xi^2}} \tag{33.14}$$

where τ_0 is a continuous function on $[-1, 1]$. According to Par. 1° of Sec. 13, the unknown $\tau_0(\xi)$ is replaced by the Lagrange interpolation polynomial (13.4):

$$L_n[\tau_0; \xi]=\frac{1}{n}\sum_{k=1}^{n}(-1)^{k+1}\tau_0(\xi_k)\frac{\cos n\vartheta \sin\vartheta_k}{\cos\vartheta-\cos\vartheta_k}, \tag{33.15}$$

$$\xi=\cos\vartheta, \qquad \xi_m=\cos\vartheta_m, \qquad \vartheta_m=\frac{2m-1}{2n}\pi \qquad (m=1, 2, \ldots, n)$$

and Eq. (33.13) is reduced to the following system of linear algebraic equations in the values $\tau_0(\xi)$ at the given nodes:

$$\sum_{k=1}^{n}\alpha_{mk}\tau_k^0=\frac{1}{2}(Q-P) \qquad (m=1, 2, \ldots, n), \tag{33.16}$$

$$\frac{\pi}{n}\sum_{k=1}^{n}\tau_k^0=-\frac{Q}{2\alpha}, \tag{33.17}$$

$$\alpha_{mk}=\frac{1}{n}\left[\frac{1}{\sin\vartheta_m}\cot\frac{\vartheta_m\pm\vartheta_k}{2}-\pi\alpha H(k-m)\right], \qquad \tau_m^0=\tau_0(\xi_m).$$

In order that the number of equations in the system agrees with the number of unknowns, it is necessary to leave out any one value of m in (33.16) and to replace the corresponding equation by Eq. (33.17).

The solution of this system of n linear equations gives, by formulas (33.14) and (33.15), an approximate solution of Eq. (33.13), which determines, as stated above, the approximate solution of the original problem. On the basis of (33.12), (33.14) and (33.15), the shearing stress intensity factors are then expressed in terms of the derived values of the function $\tau_0(\xi)$ at the interpolation nodes by the formulas

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* The kernels $k(x, t)$ of singular equations encountered in applications often have discontinuities inside and are unbounded in the domain of definition. Such are, for instance, the kernels generated by the influence function. In Eq. (33.11) the kernel, having a discontinuity of the first kind, is bounded. The following sections give examples of equations with less favourable kernels. To all these equations, however, the method of approximate solution in question is applicable.

$$\left.\begin{aligned} T_{-a}&=\sqrt{a}\,\tau_0(-1)\cong\frac{\sqrt{a}}{n}\sum_{k=1}^{n}(-1)^{k+n}\tau_k^0\tan\frac{\vartheta_k}{2},\\ T_a&=\sqrt{a}\,\tau_0(1)\cong\frac{\sqrt{a}}{n}\sum_{k=1}^{n}(-1)^{k+1}\tau_k^0\cot\frac{\vartheta_k}{2}.\end{aligned}\right\}\qquad(33.18)$$

Polynomial (33.15) is, of course, continuous on the interval $[-1, 1]$. For sufficiently large n, the preceding formulas may therefore be replaced by the following simpler ones:

$$T_{-a}\cong\sqrt{a}\,\tau_n^0,\qquad T_a\cong\sqrt{a}\,\tau_1^0.$$

It is obvious physically that the normal stress σ_x will also have a singularity near the ends of the stringer on its prolongations. The corresponding stress intensity factors are determined by the formulas

$$N_{-a}=\lim_{x\to-a}\sqrt{2|x+a|}\,\sigma_x,\qquad N_a=\lim_{x\to a}\sqrt{2(x-a)}\,\sigma_x.\quad(33.19)$$

In the first of the above formulas the left-hand limit is taken, and in the second the right-hand limit.

In representation (33.14) which we now rewrite as

$$\tau(x)=\frac{\tau^*(x)}{\sqrt{(a-x)(a+x)}}$$

it is further assumed that the function $\tau^*(x)$ satisfies the Hölder condition in the neighbourhood of both ends of the interval $[-a, a]$. On this assumption, quantities (33.19) are determined by using Muskhelishvili's formula (3.3) describing the behaviour of the Cauchy-type integral (33.5) near the ends of the line of integration.

According to the above-mentioned formula, for points x lying on the prolongation of the stringer axis to the left of the end $x=-a$ and sufficiently close to it we have

$$\sigma_x=\frac{2}{\sqrt{a}}\,\frac{\tau^*(-a)}{\sqrt{2|x+a|}}+\sigma_0(x),\qquad(33.20)$$

where $\sigma_0(x)$ denotes a function regular in the neighbourhood and admitting a singularity at $x=-a$ of order less than 1/2. On the basis of the same formula (3.3) in the right-hand neighbourhood of the point $x=a$ we have

$$\sigma_x=-\frac{2}{\sqrt{a}}\,\frac{\tau^*(a)}{\sqrt{2(x-a)}}+\sigma_0(x)\qquad(33.21)$$

with $\sigma_0(x)$ possessing the same property as in (33.20).

By virtue of (33.19) to (33.21) we now obtain

$$N_{-a}=\frac{2}{\sqrt{a}}\tau^*(-a), \qquad N_a=-\frac{2}{\sqrt{a}}\tau^*(a),$$

or, what is the same thing,

$$N_{-a}=2\sqrt{a}\,\tau_0(-1)=2T_{-a}, \qquad N_a=-2\sqrt{a}\,\tau_0(1)=-2T_a. \quad (33.22)$$

Thus, *the tensile stress intensity of the prolongation of the stringer axis is twice the shearing stress intensity under the stringer.*

System (33.16), (33.17) has been solved using an electronic computer for the case of a (unilateral) compression of the stringe ($P = 0$, $Q = -1$) for various values of the parameter α. In the computation $H(0)$ has been taken* equal to 1/2. The calculated values of coefficients (33.18) are given in a table.

Shearing Stress Intensity at Ends of Stringer
(expressed as fractions of $-Q/\sqrt{a}$)

α	3	2	1	1/2	1/3
T_{-a}	0.2308	0.2836	0.4075	0.6047	0.7812
T_a	0.0154	0.0292	0.0834	0.2176	0.3645

The system of algebraic equations has been solved for three values of the number n, viz. $n = 21, 31, 41$. It is of interest to note that the values of the factors T_{-a}, T_a are unchanged to 10^{-4} as the order of the system is increased, which is an indication of the reliability of the method used.

34. The Effect of a Stringer on Stress Distribution Near a Circular Hole

Consider an elastic plate in the form of an infinite plane with a circular hole. Suppose that the plate is reinforced with a straight stringer (elastic bar) continuously attached to it in a radial direction and extending with one end to the contour of the hole. The edge of the hole is assumed to be free from external forces. One of the bases of the stringer, namely the one which is at the hole, is acted on by an axial load and the plate is subjected at infinity to tensile forces in the direction of the stringer axis and perpendicular to it. As to the plate and the reinforcing stringer we completely retain the assumptions of Secs. 30 and 31. The plate is deformed under conditions of generalized plane stress and the elastic stringer of uniform cross section is treated as a one-dimensional continuum having no flexural rigidity.

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* In this connection see the note at the end of Sec. 34.

For simplicity, the radius of the hole is taken to be unity. We refer the surface of the plate to the plane of the variable $z = x + iy$, place the centre of the hole at the origin and locate the axis of the stringer on the segment $[1, a]$ of the real axis (Fig. 5). The axial load at the end of the stringer is denoted by p_0 and the tensile forces in the x and y directions, respectively, by P and Q.

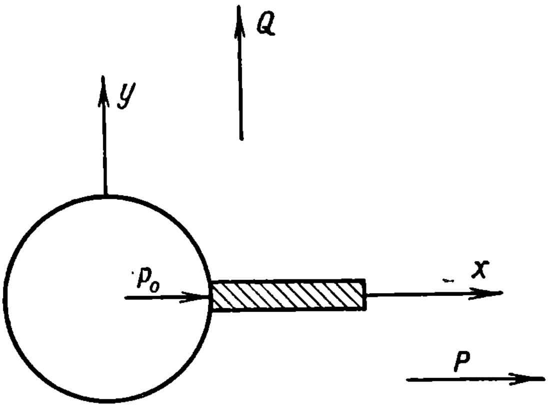

Fig. 5

We retain the previous notation for the mechanical and geometrical characteristics of the plate and stringer. In addition, we introduce the length, thickness and width of the stringer denoted, respectively, by l, h_0 and b; $l = a - 1$.

1°. **Boundary conditions.** The boundary conditions of our problem consist partly in equalities (31.6) and (31.7) which express the condition for the equilibrium of any finite part of the stringer and allow for the continuity of elastic displacements across the line of junction and the equality of extensions of the plate and stringer along the same line.

These conditions are of the form

$$\Omega^-(t, \bar{t}) - \Omega^+(t, \bar{t}) + iK_0 \operatorname{Re} \frac{d}{dt} [\varkappa\varphi^+(t) - t\overline{\varphi'^+(t)} - \overline{\psi^+(t)}] + i\frac{p_0}{h} = 0, \tag{34.1}$$

$$\varkappa\varphi^+(t) - t\overline{\varphi'^+(t)} - \overline{\psi^+(t)} = \varkappa\varphi^-(t) - t\overline{\varphi'^-(t)} - \overline{\psi^-(t)}, \quad (1 < t < a), \tag{34.2}$$

where

$$\Omega(z, \bar{z}) = \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}, \qquad K_0 = \frac{E_0 S_0}{2\mu h}, \qquad \varkappa = \frac{3-\nu}{1+\nu} \tag{34.3}$$

and $\varphi(z)$, $\psi(z)$, the complex potentials of our problem, represent sectionally holomorphic functions in the region outside the hole with the line of discontinuity L, $L = [1, a]$.

As in Sec. 31, we assume

$$\chi(z) = z\varphi'(z) + \psi(z)$$

and rewrite formula (34.2) in the form

$$\varkappa\varphi^+(t) + \overline{\chi}^+(t) = \varkappa\varphi^-(t) + \overline{\chi}^-(t). \tag{34.4}$$

By the symmetry of our problem, $\varphi(z) = \bar{\varphi}(z)$, $\psi(z) = \bar{\psi}(z)$, and hence, by formula (34.4), the function

$$\omega(z) = \varkappa\varphi(z) + z\varphi'(z) + \psi(z) \tag{34.5}$$

can be continued analytically through the interval L. The function $\omega(z)$ is therefore holomorphic in the entire region outside the circular hole.

Because of the holomorphy of $\omega(z)$ the two real conditions (34.2) reduce to the single one

$$\operatorname{Re}\{\varphi^-(t) - \varphi^+(t)\} = 0 \qquad (1 < t < a) \tag{34.6}$$

and the difference on the left-hand side of formula (34.1) is represented as

$$\Omega^-(t, \bar{t}) - \Omega^+(t, \bar{t}) = (1 - \varkappa)\operatorname{Re}\{\varphi^-(t) - \varphi^+(t)\} + \\ + i(1 + \varkappa)\operatorname{Im}\{\varphi^-(t) - \varphi^+(t)\}.$$

Equation (34.1) now falls into two (real) equations. One of these is of the form

$$(1+\varkappa)\operatorname{Im}\{\varphi^-(t)-\varphi^+(t)\}+K_0\operatorname{Re}\frac{d}{dt}[\varkappa\varphi(t)-t\overline{\varphi'(t)}-\overline{\psi(t)}]+\frac{p_0}{h}=0, \tag{34.7}$$

where the expression $\varkappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)}$ denotes the equal limiting values at the point t $(1 < t < a)$ of the function $2\mu(u + iv)$ from the left and right of L. In view of the obvious condition $1 - \varkappa \neq 0$, the second equation of (34.1) is identical with Eq. (34.6).

Thus, the boundary conditions (34.1) and (34.2) along the line of junction are equivalent to two real equalities, (34.6) and (34.7). We leave the former as it is, and take the latter in the form

$$\int_1^x (\tau_{xy}^+ - \tau_{xy}^-)\,dt + K_0\operatorname{Re}\frac{d}{dx}[\varkappa\varphi(x) - x\overline{\varphi'(x)} - \overline{\psi(x)}] + \frac{p_0}{h} = 0 \\ (1 < x < a). \tag{34.8}$$

It remains to provide a condition on the contour of the hole. Here in view of the absence of external forces, we must have

$$\varphi(\sigma) + \sigma\overline{\varphi'(\sigma)} + \overline{\psi(\sigma)} = 0 \text{ on } \gamma, \tag{34.9}$$

where γ denotes the circumference $|z| = 1$, and σ a point on it, $\sigma = e^{i\vartheta}$. Equalities (34.6), (34.8) and (34.9) constitute all conditions of our problem.

To solve the problem we make use of the influence function method presented with reference to an elastic semi-circle in Sec. 28.*

2°. **Construction of the potentials $\varphi(z)$ and $\psi(z)$.** Because of the presence of the stringer, an additional force $q(x)$ directed along the x axis arises at points x of the interval L where the stringer is attached to the plate. Its magnitude is equal to

$$q(x) = -\tau_{xy}^{+}(x, 0) + \tau_{xy}^{-}(x, 0), \tag{34.10}$$

where, it should be remembered, $\tau_{xy}^{+}\,dx$ denotes the projection on the x axis of the stress vector applied to a linear element dx adjacent to the stringer on the side of the positive y direction. Hence, at any point $z = t$ of the interval L within the plate is applied a force $(q, 0)$ and, in addition, the tensile forces P and Q are acting at infinity.

The influence functions corresponding to the given forces and denoted by $\varphi(z; t)$ and $\psi(z; t)$ are representable in the form

$$\left.\begin{aligned}\varphi(z;\ t) &= -p\ln(z-t)+\Gamma z+\varphi_0(z;\ t),\\ \psi(z;\ t) &= \varkappa p\ln(z-t)+\frac{pt}{z-t}+\Gamma' z+\psi_0(z;\ t),\end{aligned}\right\} \tag{34.11}$$

where

$$p=\frac{q}{2\pi(1+\varkappa)}, \qquad \Gamma=\frac{Q+P}{4}, \qquad \Gamma'=\frac{Q-P}{2} \tag{34.12}$$

and φ_0, ψ_0 are functions of z holomorphic for $|z|>1$ and bounded at infinity. To determine these functions we have condition (34.9) which gives

$$\left.\begin{aligned}&\varphi_0(\sigma;\ t)+\overline{\sigma\varphi_0'(\sigma;\ t)}+\overline{\psi_0(\sigma;\ t)}=f_0(\sigma;\ t),\\ &f_0(\sigma;\ t)=p\left[\ln(\sigma-t)-\varkappa\ln\left(\frac{1}{\sigma}-t\right)+\frac{\sigma(\sigma-t)}{1-\sigma t}\right|-2\Gamma\sigma-\frac{\Gamma'}{\sigma}.\end{aligned}\right\} \tag{34.13}$$

Together with the function f_0, we shall need its complex conjugate which has the form

$$\overline{f_0(\sigma;\ t)}=p\left[\ln\left(\frac{1}{\sigma}-t\right)-\varkappa\ln(\sigma-t)+\frac{1-\sigma t}{\sigma(\sigma-t)}\right]-\frac{2\Gamma}{\sigma}-\Gamma'\sigma.$$

The solution of problem (34.13) is most easily constructed by N. I. Muskhelishvili's method. The solution is given by the formulas

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* The problem of the diffusion of load from a stringer into a sheet in the case when the stringer extends with one end to the free boundary of the half-plane was investigated recently in an interesting paper by R. Muki and E. Sternberg [86]. The analysis of the problem of these authors is accompanied by a comprehensive survey of the results of a number of other works relating to the same range of problems. The paper is supplied with a rather extensive bibliography.

(see N. I. Muskhelishvili [93], Sec. 82, formulas (4′), (5′))

$$\varphi_0(z;\,t) = -\frac{1}{2\pi i}\int\limits_{\gamma}\frac{f_0\,d\sigma}{\sigma-z},\qquad \psi_0(z;\,t) = -\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\bar{f}_0\,d\sigma}{\sigma-z}-\frac{\varphi_0'(z;\,t)}{z}. \tag{34.14}$$

The above integrals are readily evaluated on the basis of the theorem and integral formula of Cauchy. For the convenience of the reader we write down all necessary formulas. We have ($|z|>1$, $1<t\leqslant a$)

$$\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\ln(\sigma-t)\,d\sigma}{\sigma-z}=0,\qquad \frac{1}{2\pi i}\int\limits_{\gamma}\frac{\ln\left(\frac{1}{\sigma}-t\right)d\sigma}{\sigma-z}=-\ln\left(1-\frac{1}{zt}\right),$$

$$\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\sigma(\sigma-t)\,d\sigma}{(1-\sigma t)(\sigma-z)}=\frac{1-t^2}{t^2(zt-1)},\qquad \frac{1}{2\pi i}\int\limits_{\gamma}\frac{1-\sigma t}{\sigma(\sigma-t)}\frac{d\sigma}{\sigma-z}=\frac{1}{zt},$$

$$\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\sigma\,d\sigma}{\sigma-z}=0,\qquad \frac{1}{2\pi i}\int\limits_{\gamma}\frac{d\sigma}{\sigma(\sigma-z)}=-\frac{1}{z}.$$

By $\ln\left(1-\frac{1}{zt}\right)$ for a fixed t is meant the single-valued branch of the logarithm which vanishes at $z=\infty$.

The preceding formulas give the holomorphic parts of the functions φ and ψ in the following form:

$$\varphi_0(z;\,t) = -p\left[\varkappa\ln\left(1-\frac{1}{zt}\right)+\frac{1-t^2}{t^2(zt-1)}\right]-\frac{\Gamma'}{z}, \tag{34.15}$$

$$\psi_0(z;\,t) = p\left[\ln\left(1-\frac{1}{zt}\right)-\frac{1}{zt}\right]-\frac{2\Gamma}{z}-\frac{\varphi_0'(z;\,t)}{z}. \tag{34.16}$$

From this, on the basis of formulas (34.11), for the influence functions φ and ψ we have, finally,

$$\varphi(z;\,t) = -p\left[\ln(z-t)+\varkappa\ln\left(1-\frac{1}{zt}\right)+\frac{1-t^2}{t^2(zt-1)}\right]+g_1(z), \tag{34.17}$$

$$\psi(z;\,t) = p\left[\varkappa\ln(z-t)+\frac{t}{z-t}+\ln\left(1-\frac{1}{zt}\right)+\frac{\varkappa}{z^2(zt-1)}-\right.$$
$$\left.-\frac{1-t^2}{zt(zt-1)^2}-\frac{1}{zt}\right]+g_2(z), \tag{34.18}$$

$$g_1(z) = -\frac{\Gamma'}{z}+\Gamma z,\qquad g_2(z) = -\frac{2\Gamma}{z}+\Gamma'\left(z-\frac{1}{z^3}\right). \tag{34.19}$$

Assume now ($L=[1,\,a]$)

$$\left.\begin{aligned}\varphi(z) &= \int\limits_{L}\varphi(z;\,t)\,dt+(1-l)\,g_1(z),\\ \psi(z) &= \int\limits_{L}\psi(z;\,t)\,dt+(1-l)\,g_2(z).\end{aligned}\right\} \tag{34.20}$$

Rewrite these formulas in a form more convenient for subsequent application

$$\left.\begin{aligned} \varphi(z) &= \frac{1}{2\pi}\int\limits_L \Omega_1(z;\ t)\,\tau(t)\,dt + g_1(z), \\ \psi(z) &= \frac{1}{2\pi}\int\limits_L \Omega_2(z;\ t)\,\tau(t)\,dt + g_2(z), \end{aligned}\right\} \tag{34.21}$$

$$\left.\begin{aligned} \Omega_1(z;\ t) &= -\ln(z-t) - \varkappa\ln\left(1-\frac{1}{zt}\right) - \frac{1-t^2}{t^2(zt-1)}, \\ \Omega_2(z;\ t) &= \varkappa\ln(z-t) + \frac{t}{z-t} + \ln\left(1-\frac{1}{zt}\right) + \\ &\qquad + \frac{\varkappa}{z^2(zt-1)} - \frac{1-t^2}{zt(zt-1)^2} - \frac{1}{zt}, \end{aligned}\right\} \tag{34.22}$$

$$\tau(x) = 2\pi p(x). \tag{34.23}$$

From formulas (34.23), (34.10), (34.12) it follows that the unknown density function $\tau(x)$ of the foregoing integrals is related to the forces along the stringer by the equation

$$-(1+\varkappa)\,\tau(x) = \tau_{xy}^{+}(x,\ 0) - \tau_{xy}^{-}(x,\ 0). \tag{34.24}$$

The functions (34.21) represent the complex potentials of our problem. Note that in the case when there is no stringer, $\tau(x) = 0$ on L, the integral terms in formulas (34.21) vanish and we have the functions φ and ψ left which give the well-known solution of the problem of stresses in a plane with a force-free circular hole under tension in two mutually perpendicular directions.

3°. **Reduction to an integral equation.** The potentials φ and ψ, as seen from their very construction, satisfy the boundary condition (34.9) for any τ. It is easily verified that condition (34.6) is also satisfied. It remains to satisfy the single condition (34.8).

For this purpose we set up the kernel of the expression $-\varkappa\varphi(z) + \overline{z\varphi'(z)} + \overline{\psi(z)}$ for $z = x$. From formulas (34.22) we find after elementary computations

$$\begin{aligned} \Omega(x,\ t) &= \frac{1}{2\pi}\left[-\varkappa\Omega_1(z;\ t) + z\overline{\Omega_1'(z;\ t)} + \overline{\Omega_2(z;\ t)}\right]_{z=x} = \\ &= \frac{1}{2\pi}\left\{\varkappa\left[\ln(x-t) + \overline{\ln(x-t)}\right] + (\varkappa^2+1)\ln\left(1-\frac{1}{xt}\right) + \right. \\ &\quad \left. + \varkappa\left[\frac{1-t^2}{t^2} - \frac{1-x^2}{x^2}\right]\frac{1}{xt-1} + \frac{(x^2-1)(1-t^2)}{xt(xt-1)^2} - \frac{1+xt}{xt}\right\}. \end{aligned} \tag{34.25}$$

On differentiating the above expression with respect to x the terms containing the logarithms give discontinuous functions of two variables, Cauchy-type kernels. The remaining terms represent regular functions of x $(x > 1,\ t > 1)$ and their differentiation yields no singular terms.

To calculate the singular terms use should be made of the formulas

$$\left.\begin{aligned} \frac{d}{dx}\frac{1}{\pi i}\int_L \ln(x-t)\,\tau(t)\,dt &= -\tau(x)+\frac{1}{\pi i}\int_L \frac{\tau(t)\,dt}{x-t}, \\ \frac{d}{dx}\frac{1}{\pi i}\int_L \overline{\ln(x-t)}\,\tau(t)\,dt &= \tau(x)+\frac{1}{\pi i}\int_L \frac{\tau(t)\,dt}{x-t} \end{aligned}\right\} \qquad (34.26)$$

which are valid whenever the function $\tau(x)$ is Hölder continuous on the interval $(1, a)$.

In accordance with the above we obtain

$$\frac{d}{dx}\int_L \Omega(x, t)\,\tau(t)\,dt = \frac{1}{2\pi}\int_L \left\{\frac{2\varkappa}{x-t}+\frac{\partial}{\partial x}K(x, t)\right\}\tau(t)\,dt, \qquad (34.27)$$

where

$$K(x, t) = (\varkappa^2+1)\ln\left(1-\frac{1}{xt}\right)-\varkappa\left(\frac{t^2-1}{t^2}+\frac{x^2-1}{x^2}\right)\frac{1}{xt-1}- \\ -\frac{(x^2-1)(t^2-1)}{xt\,(xt-1)^2}-\frac{1+xt}{xt}. \qquad (34.28)$$

We now insert expressions (34.21) in the boundary condition (34.8). The difference $\tau_{xy}^{+}-\tau_{xy}^{-}$ under the integral in the first term of the equality is replaced by τ according to (34.24), and to calculate the second term use is made of formula (34.27), taking also into account notation (34.19). As a result of very simple computations we arrive at the following relation:

$$\frac{1}{2\pi}\int_L \frac{\tau(t)\,dt}{t-x}+\frac{1}{2\pi}\int_L k_0(x, t)\,\tau(t)\,dt = f(x), \qquad (34.29)$$

where

$$k_0(x, t) = -\frac{1}{2\varkappa}\frac{\partial}{\partial x}K(x, t)-2\pi\lambda_0 H(x-t), \qquad (34.30)$$

$$f(x) = -\frac{\Gamma}{2\varkappa}\left(\varkappa-1-\frac{2}{x^2}\right)+\frac{\Gamma'}{2\varkappa}\left(1-\frac{\varkappa+1}{x^2}+\frac{3}{x^4}\right)-\frac{\mu p_0}{\varkappa S_0 E_0}, \qquad (34.31)$$

$$\lambda_0 = \frac{(\varkappa+1)\,\mu h}{\varkappa S_0 E_0}, \qquad (34.32)$$

$$H(u) = 1, \quad (u > 0), \quad H(u) = 0 \quad (u < 0). \qquad (34.33)$$

Relation (34.29) represents a singular integral equation of the first kind. It serves to determine the density function $\tau(x)$ of the complex potentials.*

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* In nearly all investigations devoted to a study of the stringer-plate model the problem posed is that of determining the single function $\tau(x)$, while the question of the stresses in the plate outside the interval L is not generally discussed. As appears from the above (see also the next paragraph), all elements of the elastic fields in the plate and stringer may be expressed in terms of $\tau(x)$ by means of definite integral operators which make it possible to bring the solution to numerical results.

4°. **Computational scheme. Numerical results.** By a linear transformation

$$z = \alpha\zeta + \beta, \quad \alpha = \frac{l}{2}, \qquad \beta = \frac{l}{2} + 1, \tag{34.34}$$

where l is the length of the stringer, the interval $L = [1, a]$ is mapped into an interval $[-1, 1]$ in the plane $\zeta = \xi + i\eta$ and the circle $|z| = 1$ is transformed into a circle of radius $2/l$ with centre at the point $\zeta = -1 - \frac{2}{l}$. Equation (34.29) then changes to

$$\frac{1}{2\pi}\int_{-1}^{1} \frac{\tau(\eta)\,d\eta}{\eta-\xi} + \frac{1}{2\pi}\int_{-1}^{1} k(\xi, \eta)\,\tau(\eta)\,d\eta = f(\xi) \qquad (-1<\xi<1), \tag{34.35}$$

where, to avoid introducing too many symbols, use has been made of the notation

$$k(\xi, \eta) = k_1(\xi, \eta) - 2\pi\lambda H(\xi-\eta), \tag{34.36$'$}$$

$$\begin{aligned} k_1(\xi, \eta) = -\frac{1}{2\varkappa}\frac{\partial}{\partial\xi} K(\xi, \eta) = -\frac{\alpha}{2\varkappa}\Big\{ &\left(\varkappa^2 + 1 - \frac{2\varkappa}{x^2}\right)\frac{1}{x(xt-1)} + \\ &+ \left[\varkappa xt\left(\frac{t^2-1}{t^2} + \frac{x^2-1}{x^2}\right) - \frac{(t^2-1)(x^2+1)}{xt}\right]\frac{1}{x(xt-1)^2} + \\ &+ \frac{2(t^2-1)(x^2-1)}{x(xt-1)^3} + \frac{1}{x^2t}\Big\}, \end{aligned} \tag{34.36$''$}$$

$$\tau(x) = \tau(\xi), \qquad K(x, t) = K(\xi, \eta), \qquad f(x) = f(\xi), \tag{34.37}$$

$$x = \alpha\xi + \beta, \qquad t = \alpha\eta + \beta, \qquad \lambda = \alpha\lambda_0. \tag{34.38}$$

The kernel $k(\xi, \eta)$ in Eq. (34.35) consists of two terms. The first of these, as formula (34.36″) shows, is a continuous function of ξ and η in the closed square $-1 \leqslant \xi, \eta \leqslant 1$ from which a vertex is removed, located at the lower left corner ($\xi = \eta = -1$). At this latter point the first term has a (fixed) singularity such as in Cauchy-type kernels. The second term, because of the Heaviside step function H occurring in it, undergoes a finite discontinuity at $\xi = \eta$.

If these features of the function $k(\xi, \eta)$ are disregarded, Eq. (34.29) will belong to the class of singular equations considered in Sec. 13.

Below the method indicated in Sec. 13 is applied to the solution of Eq. (34.35). We take for granted that there exists a solution $\tau(\xi)$ for Eq. (34.35) belonging on $[-1, 1]$ to the class H^* (see Sec. 3, Par. 1°). This means that the function $\tau(\xi)$ satisfies the Hölder condition in any closed part of the interval $[-1, 1]$ not containing the ends, and near the ends it may become infinite with degree less than unity.

As to the behaviour of the solution $\tau(\xi)$ in the neighbourhood of the ends of the interval we can make more exact assumptions. By considering a semi-infinite stringer attached to a solid infinite plate and stretched by an axial load, it has been established (E. L. Buell [15], W. T. Koiter [59]) that the tangential force $\tau_{xy}^{+}(x, 0)$ has a singularity of order 1/2 at the end of the stringer (lying in a finite part of the plane). By analogy, it seems obvious that the unknown function $\tau(x)$ must have the same order of singularity at the right end of the interval $L = [1, a]$, i.e., in the neighbourhood of $x = a$ it must behave as $O((a - x)^{-1/2})$. At the left end of the interval L where the stringer meets the free boundary at right angles, the singularity of the forces $\tau(x)$ must be weaker from physical considerations.*

To Eq. (34.29), as in the preceding section, is added one more condition, namely the condition of equilibrium of the attached stringer

$$\int_L \tau(t)\,dt = \frac{p_0}{(1+\varkappa)h} \tag{34.39}$$

which we take in the form

$$\int_{-1}^{1} \tau(\eta)\,d\eta = A, \qquad A = \frac{p_0}{(1-\varkappa)h\alpha}. \tag{34.40}$$

According to the considerations developed above, the solution of Eq. (34.35) admits representation (33.14) with a continuous function $\tau_0(\xi)$ on the interval which vanishes at the left end of the interval.

The application of the procedure described in the preceding section reduces the solution of (34.35), (34.40) to the system of linear equations

$$\left.\begin{aligned} &\sum_{\nu=1}^{n} \alpha_{m\nu}\tau_\nu^0 = f_m \quad (m = 1, 2, \ldots, n), \\ &\frac{\pi}{n}\sum_{\nu=1}^{n} \tau_\nu^0 = A \end{aligned}\right\} \tag{34.41}$$

with

$$\alpha_{m\nu} = \frac{1}{2n}\left[\frac{1}{\sin\vartheta_m}\cot\frac{\vartheta_m \pm \vartheta_\nu}{2} + k(\xi_m, \eta_\nu)\right], \tag{34.42}$$

$$f_m = f(\xi_m), \qquad \tau_m^0 = \tau_0(\xi_m), \qquad \xi_m = \eta_m = \cos\vartheta_m.$$

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* In the above-cited paper by R. Muki and E. Sternberg a characteristic equation is derived which determines the order of singularity of the forces $\tau(x)$ at the edge of a stringer for the case when the loaded end of the stringer is at a free half-plane boundary ([86], Eq. (32)).

The same result can be obtained from somewhat different considerations based on the general investigations by I. I. Vorovich of the behaviour of elastic solutions near corners.

According to the calculations presented in the afore-mentioned work of two authors, the singularity of the stresses $\tau(x)$ (near the loaded end of the stringer) is of order 0.25 for $\nu = 0.3$.

The functions $f(\xi)$ and $K(\xi, \eta)$ are determined by equalities (34.31) and (34.36′). It may be recalled that in formula (34.42) the upper sign is taken in the case when the number $|m-\nu|$ is even, and the lower sign when it is odd; zero is considered as an even number.

After finding the solution of (34.41), the required approximate solution of Eq. (34.35) is represented in the form of (34.14), where by $\tau_0(\xi)$ is meant polynomial (34.15) formed with the numbers τ_ν^0.

We now turn to the calculation of the physical quantities. Our interest will be centred on the stress field near the hole. In order to have an idea of the effect of the stringer on this field, it is necessary to determine the values of the tensile stress σ_ϑ at points of the hole contour characteristic for the given external force.

Referring to the well-known formula

$$\sigma_\vartheta + \sigma_r = 4 \operatorname{Re} \varphi'(z), \tag{34.43}$$

calculate its right-hand side* for $z = \pm i, -1$.

From the first formula of (34.21) we obtain by differentiation

$$\varphi'(z) = \frac{1}{2\pi} \int_L \left[\frac{1}{t-z} - \frac{\varkappa}{z(zt-1)} + \frac{1-t^2}{t(zt-1)^2} \right] dt + \frac{\Gamma'}{z^2} + \Gamma.$$

The function $\varphi'(z)$ is obviously everywhere holomorphic for $|z| > 1$, with the exception of the points of the interval L. It is easily verified that at the points $z = \pm i$ the function $\varphi'(z)$ has the same real parts and their common value is

$$\operatorname{Re} \varphi'(z) = \frac{1}{2\pi} \int_L M(t)\, \tau(t)\, dt + \Gamma - \Gamma' \qquad (z = \pm i), \tag{34.44}$$

$$M(t) = \frac{(\varkappa+1)t}{t^2+1} + \frac{(t^2-1)^2}{t(t^2+1)^2}.$$

The function $M(t)$ is continuous on the interval L. To evaluate the integral on the right-hand side of (34.44), it is therefore advisable to use the quadrature formula (13.8) which, with (33.14), gives

$$\frac{1}{2\pi} \int_L M(t)\, \tau(t)\, dt = \frac{\alpha}{2\pi} \int_{-1}^{1} M(\eta)\, \tau(\eta)\, d\eta = \frac{l}{n} \sum_{k=1}^{n} \tau_k^0 M(\eta_k),$$

where, in accordance with the notation rule repeatedly applied above,

$$M(\eta) = M(t) = M(\alpha\eta + \beta), \qquad \eta_k = \cos \vartheta_k.$$

Calculate also the value of $\varphi'(z)$ for $z = -1$. We remark first that the function $\varphi'(z)$ takes real values on the x axis outside the interval L. Its value at $z = -1$ is

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* It will be recalled that, by the condition of the problem, $\sigma_r = 0$ on the hole contour.

$$\varphi'(z) = \frac{1}{2\pi}\int_L N(t)\,\tau(t)\,dt + \Gamma + \Gamma' \qquad (z = -1),$$

$$N(t) = -\left(\varkappa - \frac{1}{t}\right)\frac{1}{t+1}.$$

This integral is evaluated in the same way as the preceding one and is approximately equal to

$$\frac{1}{2\pi}\int_L N(t)\,\tau(t)\,dt = \frac{l}{n}\sum_{k=1}^{n} \tau_k^0 N(\eta_k).$$

Of interest is also the shearing stress intensity factor near the end $x = a$ which is defined by the formula

$$T_a = \lim_{x\to a}\sqrt{2(a-x)}\,[\tau_{xy}^{+}(x, 0) - \tau_{xy}^{-}(x, 0)].$$

On the basis of (34.24) the above formula may be written as

$$-T_a = (1+\varkappa)\lim_{x\to a}\sqrt{2(a-x)}\,\tau(x).$$

As in the preceding section, T_a is determined in terms of the solution of system (34.41) as follows:

$$-T_a = (1+\varkappa)\sqrt{\alpha}\,\tau_0(1) \cong (1+\varkappa)\sqrt{\alpha}\,\frac{1}{n}\sum_{k=1}^{n}(-1)^{k-1}\tau_k^0\cot\frac{\vartheta_k}{2}. \tag{34.45}$$

For sufficiently large n, this formula may be replaced by a simpler one

$$-T_a = (1+\varkappa)\sqrt{\alpha}\,\tau_1^0 \qquad \left(\alpha = \frac{l}{2}\right).$$

The numerical solution has been obtained in the case of the extension of a plate in the x direction when

$$P = 1, \qquad p_0 = Q = 0, \qquad \Gamma = \frac{1}{4}, \qquad \Gamma' = -\frac{1}{2}.$$

The right-hand side of (34.43) has been calculated at the point $z = i$ for various values of the parameter

$$g = \frac{E_0 h_0}{Eh}$$

and the length l. The computation has been carried out for $\nu = 1/3$, $b = 0.2$ (ν is Poisson's ratio). The numerical results are given in a table.

Values of $\bar{\sigma}_\vartheta = \sigma_\vartheta - 3$ at Points $z = \pm i$ and of $\bar{\tau} = -(1+\varkappa)\tau_1^0$

	l \ g	1/5	1/3	1	3	5
$\bar{\sigma}_\vartheta$	1	2.83×10^{-4}	4.35×10^{-4}	9.65×10^{-4}	1.66×10^{-3}	1.95×10^{-2}
$\bar{\sigma}_\vartheta$	2	1.60×10^{-3}	2.57×10^{-3}	6.55×10^{-3}	1.38×10^{-2}	1.77×10^{-2}
$\bar{\tau}$	1	0.101	0.128	0.198	0.270	0.297
$\bar{\tau}$	2	0.117	0.151	0.250	0.378	0.438
	g \ l					7
$\bar{\sigma}_\vartheta$	1	0.0066	0.0189	0.0268	0.0317	0.0350
$\bar{\tau}$	1	0.2495	0.2288	0.1991	0.1765	0.1594

It will be recalled for comparison that the value of σ_ϑ at the same point, with no stringer, is 3.

Note. From the way of writing the basic equation in the form of (34.29), the diagonal terms in system (34.41) involve the Heaviside function H with a zero argument (it appears in the last term on the right-hand side of formula (34.42) with $m = \nu$). In the computation $H(0)$ has been taken equal to 1/2, which is equivalent to applying the trapezoidal rule for the quadrature of the integral

$$\int_{\vartheta_m}^{\pi} \tau_0(\cos\vartheta)\, d\vartheta.$$

From the foregoing consideration it will appear that the accuracy of the quadrature formulas employed above may not always be high. To obtain acceptable numerical results, it is necessary, therefore, to take the number n in system (34.41) sufficiently large (of order 30 or 40).

35. The Case of Symmetrical Reinforcement

The case of a symmetrically reinforced plate when a stringer of length l is also attached to the segment $[-a, -1]$ of the real axis and, when there is force symmetry, is treated in a similar way.

Suppose that the right base of the stringer situated over the segment $[-a, -1]$ is acted on by an opposite axial load p_0 and the medium is subjected, as before, to biaxial tension at infinity.

The condition of equilibrium of any elementary volume of the stringer here gives

$$h\int_{x_0}^{x}(\tau_{xy}^{+}-\tau_{xy}^{-})\,dt+N(x)+p_0=0,\qquad x_0=\begin{cases}1, & 1\leqslant x\leqslant a,\\ -1, & -a\leqslant x\leqslant -1,\end{cases}\tag{35.1}$$

$$\sigma_y^{+}-\sigma_y^{-}=0\qquad (1<|x|<a).\tag{35.2}$$

Because of the symmetry of the problem there are the following physically obvious relations:

$$q(-x)=-q(x),\qquad N(-x)=N(x)\qquad (1<x<a),\tag{35.3}$$

where q stands for (34.10). By virtue of (35.3) equalities (35.1) are identical, and so we may limit ourselves in what follows to either one of these; for definiteness, we shall consider the first equality.

The continuity conditions (31.3), which are, of course, also valid here, together with (35.1) and (35.2), lead to the following conditions for the complex potentials:

$$\left.\begin{aligned}&\operatorname{Re}\{\varphi^{-}(x)-\varphi^{+}(x)\}=0\qquad (1<|x|<a),\\ &\int_{1}^{x}(\tau_{xy}^{+}-\tau_{xy}^{-})\,dt+K_0\operatorname{Re}\frac{d}{dx}\left[\varkappa\varphi(x)-x\overline{\varphi'(x)}-\overline{\psi(x)}\right]+\frac{p_0}{h}=0\\ &\qquad\qquad (1\leqslant x\leqslant a).\end{aligned}\right\}\tag{35.4}$$

To the above equalities must be added (34.9), the condition for zero external forces along γ.

The forces $q(x)$ in our case arise at points of both intervals, $(-a, -1)$, $(1, a)$, and are governed by the first of conditions (35.3). On this basis it can easily be verified that the influence functions for the problem under consideration are of the form

$$\left.\begin{aligned}\varphi(z;\ t)&=\Omega_1^{*}(z;\ t)\,p(t)+g_1(z),\\ \psi(z;\ t)&=\Omega_2^{*}(z;\ t)\,p(t)+g_2(z),\end{aligned}\right\}\tag{35.5}$$

$$\Omega_j^{*}(z;\ t)=\Omega_j(z;\ t)-\Omega_j(z;\ -t)\qquad (j=1,\ 2),$$

where Ω_1 and Ω_2 are defined by formulas (34.22) and g_1, g_2 have the former values.

From (35.5), by multiplying both sides of the equalities by dt and integrating along $L=[1, a]$, we find a representation of the complex potentials:

$$\left.\begin{aligned}\varphi(z) &= \frac{1}{2\pi}\int_L \Omega_1^*(z;\ t)\,\tau(t)\,dt + g_1(z),\\ \psi(z) &= \frac{1}{2\pi}\int_L \Omega_2^*(z;\ t)\,\tau(t)\,dt + g_2(z)\end{aligned}\right\} \qquad (35.6)$$

which reduces the problem to a singular equation. This equation is of the form

$$\frac{1}{2\pi}\int_L \left(\frac{1}{t-x}+\frac{1}{t+x}\right)\tau(t)\,dt + \frac{1}{2\pi}\int_L k_0^*(x,\ t)\,\tau(t)\,dt = f(x), \qquad (35.7)$$

$$k_0^*(x,\ t) = -2\pi\lambda_0 H(x-t) - \frac{1}{2\varkappa}\frac{\partial}{\partial x}\left[K(x,\ t) - K(x,\ -t)\right],$$

where λ_0, f and K are determined by the formulas of the preceding section. To the above equation must, of course, be added Eq. (34.39) which expresses the condition for equilibrium of each stringer separately.

Formula (34.44) in our case takes the form

$$\operatorname{Re}\varphi'(z) = \frac{1}{\pi}\int_L M(t)\,\tau(t)\,dt + \Gamma - \Gamma' \qquad (z = \pm i),$$

where $M(t)$ has the previous meaning.

All the rest is unchanged.

36. Application to Crack Problems

The influence function method developed in Sec. 28 may be used to advantage in crack problems. This statement will be illustrated

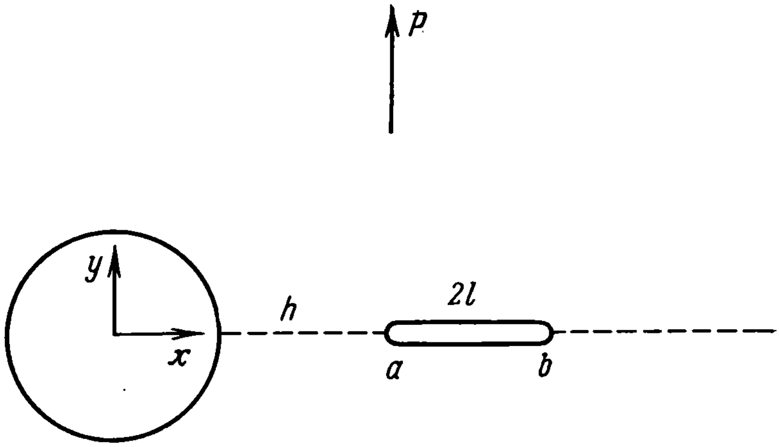

Fig. 6

by several examples whose geometry is shown in Figs. 6 and 7. It is easily verified that they are less complicated than the problems of the preceding sections also based on the method of Sec. 28.

An elastic medium occupies the entire plane $z = x + iy$ with a circular hole of unit radius and a radial cut of length $2l$ situated on the x axis. We shall consider simultaneously two cases: when the hole is filled with an absolutely rigid core bonded to the matrix along its contour, and when the hole is not filled with anything (circular cavity) and the contour is free from external forces. In both cases it is assumed that the faces of the cut are free from external forces, the medium is subjected at infinity to tensile forces P perpendicular to the line of the cut and there is no rotation in infinitely remote parts of the plane. The contour of the hole is denoted, as before, by γ and the cut, without its ends, by L. We distinguish the upper and lower sides of the cut and assign to the associated quantities a plus and a minus sign, respectively.

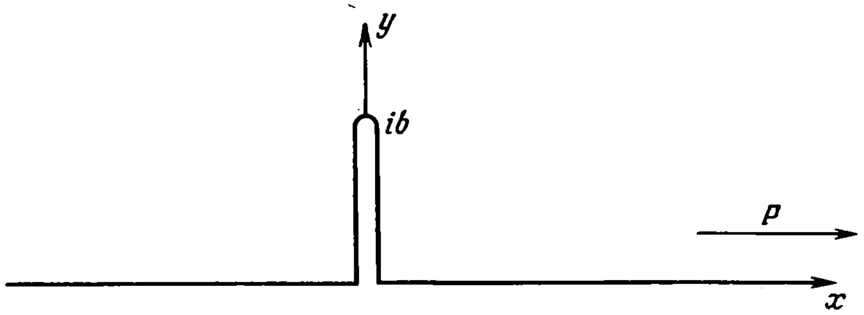

Fig. 7

The boundary conditions of the problem are

$$u = v = 0 \text{ or } \sigma_r = \tau_{r\theta} = 0 \text{ on } \gamma, \tag{36.1}$$

$$\sigma_y^{\pm} = \tau_{xy}^{\pm} = 0 \text{ on } L. \tag{36.2}$$

Equalities (36.1) express the condition for zero elastic displacements along the contour of the hole or the condition for zero external forces. The significance of condition (36.2) is obvious.

Introduce the notation

$$\left.\begin{aligned} &\omega(z) = \frac{\varphi'(z)}{z} + \psi(z), \\ &\Omega(z, \bar{z}) = \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} = \\ &\qquad = \varphi(z) + \overline{\omega(z)} + \left(z - \frac{1}{\bar{z}}\right)\overline{\varphi'(z)}. \end{aligned}\right\} \tag{36.3}$$

Then, according to representations (1.2), (1.3) which we rewrite as

$$2\mu(u + iv) = (\varkappa + 1)\varphi(z) - \Omega(z, \bar{z}), \tag{36.4}$$

$$\left.\begin{aligned} &\sigma_x + \sigma_y = 2[\varphi'(z) + \overline{\varphi'(z)}], \\ &\sigma_y - \sigma_x + 2i\tau_{xy} = 2\frac{\partial}{\partial \bar{z}}\Omega(z, \bar{z}) \end{aligned}\right\} \tag{36.5}$$

the boundary conditions of the problem take the form

$$\lambda\varphi(\sigma) + \overline{\omega(\sigma)} = 0 \text{ on } \gamma \qquad (\sigma = e^{i\vartheta}), \tag{36.6}$$

$$\sigma_y^{\pm} - i\tau_{xy}^{\pm} = \frac{d}{dx}\Omega(x, x) = 0 \text{ on } L. \tag{36.7}$$

Here $\lambda = 1$ in the case of the cavity and $\lambda = -\varkappa$ in the case of the rigid inclusion; $\varkappa = 3 - 4\nu$ for plane strain, ν is Poisson's ratio.

For convenience the required solution is represented as the sum

$$\left.\begin{aligned} &\varphi(z) = \varphi_*(z) + \varphi_0(z), \\ &\omega(z) = \omega_*(z) + \omega_0(z) \qquad (\psi(z) = \psi_*(z) + \psi_0(z)), \end{aligned}\right\} \tag{36.8}$$

where φ_0, ψ_0 give the solution of the problem with no cut, and φ_*, ψ_* define the additional field caused by the cut.

The functions φ_0, ω_0 are easily determined from condition (36.6) by Muskhelishvili's method. The solution of the plane problem for the outside of the circle

$$\lambda\varphi(\sigma) + \omega(\sigma) = f(\sigma) \text{ on } \gamma, \tag{36.9}$$

subject to the condition that the stresses and rotation vanish at infinity and the resultant of the external forces applied to the boundary of the elastic body is zero, is given by the formulas (N. I. Muskhelishvili [93])

$$\left.\begin{aligned} &\lambda\varphi(z) = \frac{1}{2\pi i}\int_{\gamma}\frac{f(\sigma)\,d\sigma}{\sigma - z}, \\ &\omega(z) = \frac{1}{2\pi i}\int_{\gamma}\frac{\overline{f(\sigma)}\,d\sigma}{\sigma - z} + \text{constant} \qquad (|z| > 1). \end{aligned}\right\} \tag{36.10}$$

By using these formulas and noting that for large $|z|$

$$\varphi_0(z) = Az + \varphi_0^*(z), \qquad \omega_0(z) = Bz + \omega_0^*(z),$$

$$A = \frac{P}{4}, \qquad B = \frac{P}{2},$$

where φ_0^*, ω_0^* are regular functions everywhere outside the circular hole, we find, rejecting the non-essential constant in (36.10),

$$\varphi_0(z) = Az - \frac{B}{\lambda z}, \qquad \omega_0(z) = Bz - \frac{\lambda A}{z} \qquad (|z| > 1). \tag{36.11}$$

The finding of the functions φ_*, ω_* is much more complicated; it is equivalent to solving the problem.

From the boundary condition (36.6) we determine functions $\varphi_*(z; t)$, $\omega_*(z; t)$ dependent on the point t from L, analytic in z, regular everywhere outside the hole, with the exception of the points of the interval L, and admitting in the neighbourhood of $z = t$

the representations (influence functions)

$$\left.\begin{aligned} \varphi_*(z;\ t) &= -p(t)\ln(z-t)+\varphi_*^0(z;\ t),\\ \omega_*(z;\ t) &= -p(t)\left[\ln(z-t)+\frac{1-zt}{z(z-t)}\right]+\omega_*^0(z;\ t). \end{aligned}\right\} \qquad (36.12)$$

Here $p(t)$ is some, as yet arbitrary, real function of the point t on L.

The functions φ_*^0, ω_*^0 must be regular everywhere for $|z|>1$, including the infinitely remote point of the plane, and for any t from L. For their determination we have, by (36.6),

$$\left.\begin{aligned} &\lambda\varphi_*^0(\sigma;\ t)+\overline{\omega_*^0(\sigma;\ t)}=f_0(\sigma;\ t),\\ &f_0(\sigma;\ t)=p(t)\left[\lambda\ln(\sigma-t)+\ln\left(\frac{1}{\sigma}-t\right)+\frac{\sigma(\sigma-t)}{1-\sigma t}\right]. \end{aligned}\right\} \qquad (36.13)$$

Thus, we have again arrived at problem (36.9). Solving it by the same formulas (36.10) gives

$$\left.\begin{aligned} \lambda\varphi_*^0(z;\ t) &= p(t)\left[\ln\left(1-\frac{1}{zt}\right)-\frac{1-t^2}{t^2(zt-1)}\right],\\ \omega_*^0(z;\ t) &= p(t)\left[\lambda\ln\left(1-\frac{1}{zt}\right)-\frac{1}{zt}\right]. \end{aligned}\right\} \qquad (36.14)$$

On the basis of (36.12) and (36.14) we now find a representation of the potentials of the additional field in the following form:

$$\left.\begin{aligned} \varphi_*(z) &= -\frac{1}{\pi}\int_L K_1(z,\ t)\,p(t)\,dt,\\ \omega_*(z) &= -\frac{1}{\pi i}\int_L K_2(z,\ t)\,p(t)\,dt, \end{aligned}\right\} \qquad (36.15)$$

where

$$\left.\begin{aligned} K_1(z,\ t) &= \ln(z-t)-\frac{1}{\lambda}\left[\ln\left(1-\frac{1}{zt}\right)-\frac{1-t^2}{t^2(zt-1)}\right],\\ K_2(z,\ t) &= \ln(z-t)+\frac{1-zt}{z(z-t)}-\lambda\ln\left(1-\frac{1}{zt}\right)+\frac{1}{zt}. \end{aligned}\right\} \qquad (36.16)$$

According to (36.8), the potentials $\varphi(z)$ and $\omega(z)$ in the resultant field are obtained by adding the two component potentials, (36.11) and (36.15).

To clarify the mechanical significance of the function $p(x)$, consider the expression

$$K(x,\ t)=K_1(z,\ t)+\overline{K_2(z,\ t)}+\left(z-\frac{1}{z}\right)\overline{\frac{\partial K_1(z,\ t)}{\partial z}} \qquad (z=x).$$

From formulas (36.16) we find

$$K(x,\ t)=\ln(x-t)+\overline{\ln(x-t)}+K_0(x,\ t), \qquad (36.17)$$

$$K_0(x,\ t)=-\frac{\lambda^2+1}{\lambda}\ln\left(1-\frac{1}{xt}\right)+\left(\frac{1-t^2}{t^2}+\frac{1-x^2}{x^2}\right)\frac{1}{\lambda(xt-1)}+ \\ +\frac{(1-t^2)(1-x^2)}{\lambda xt(xt-1)^2}+\frac{xt+1}{xt}. \qquad (36.18)$$

From the above formulas and representation (36.15) it follows that the combination

$$\Omega_* (z, \bar{z}) = \varphi_* (z) + z\overline{\varphi_*' (z)} + \overline{\psi_* (z)}$$

is real on the x axis, i.e.,

$$\operatorname{Im} \Omega_* (x, x) = 0 \qquad (|x| \geqslant 1). \tag{36.19}$$

By differentiating (34.4) with respect to x, we find

$$2\mu \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) = (\varkappa + 1) \varphi' (z) - \frac{\partial}{\partial x} \Omega (z, \bar{z}). \tag{36.20}$$

Potentials (36.11) are real on the x axis, and in consequence the vertical displacement v and the shearing stress τ_{xy} corresponding to these potentials are everywhere zero on the x axis. Hence, from (36.19) and (36.20) it follows that

$$2\mu \frac{\partial v}{\partial x} = (\varkappa + 1) \operatorname{Im} \varphi_*' (z) \text{ for } z = x \qquad (|x| > 1, \ x \neq a, b).$$

From this and from (36.15), by using the Sokhotskii-Plemelj formulas, we obtain for points x lying on the cut L

$$\pm p(x) = \frac{2\mu}{\varkappa + 1} \frac{dv^{\pm}}{dx} \quad \text{on} \quad L. \tag{36.21}$$

Note. The idea of the method used here is the construction in explicit form of potentials (36.15) corresponding to unknown normal displacements distributed along the faces of the cut. As will be shown later, these potentials reduce the problem to a singular equation in the unknown dislocation density $p(x)$.

The method has been suggested by H. F. Bueckner [12]. Subsequently, it has been developed mainly by H. F. Bueckner himself and used in a series of works devoted to the determination of the stress field intensity in the neighbourhood of crack tips (see, for example, H. F. Bueckner and I. Giaever [13], H. F. Bueckner [14]). Basically much the same method is contained in the author's paper [46] on the bending of semi-circular plates published almost concurrently with that of H. F. Bueckner [12].

It should be noted that the representation density $p(x)$ may have different physical meaning. In the above-mentioned paper of the present author the function $p(x)$ is defined on the interval (—1, 1), complex-valued and occurs as a generalized load concentrated at a point of the interval. In the case considered, as appears from formula (36.21), the integral of the function $\frac{1+\varkappa}{2\mu} p(x)$ on the interval $[a, x]$ is interpreted as a normal displacement, also "concentrated" at a point of the cut L. In H. F. Bueckner's potentials, however, the *density itself*, apart from the same factor, has the meaning of a "concentrated" normal displacement. These latter will be used in the next section.

Note, finally, that if there is no cut, then $v^+ = v^- =$ constant on L, potentials (36.15) vanish and formulas (36.8), (36.11) give the solution of the well-known problem of the simple extension of a plate with a rigid circular inclusion or a circular cavity.

Revert to our problem. Potentials (36.11), (36.15) satisfy, by construction, the boundary condition on γ for any $p(x)$. Also, as already mentioned, they are real for a real z, and hence, by (36.5), they give no shearing stresses different from zero on the x axis, and in particular along the cut. Consequently, it remains to satisfy only the first of conditions (36.2) on L which, in accordance with (36.7), takes the form

$$\frac{d}{dx}[\Omega_*(x, x) + \Omega_0(x, x)] = 0 \quad \text{on} \quad L, \tag{36.22}$$

where Ω is defined by the second formula of (36.3) and the functions φ, ω occurring in it are given by (36.11) and (36.15). In other words, in order to satisfy all conditions of the problem, it is sufficient to equate to zero the calculated normal stress σ_y for the points of the cut in the resultant field.

On the basis of the above formulas equality (36.22) is written as

$$\frac{d}{dx}\left\{-\frac{1}{\pi}\int_L K(x, t)\, p(t)\, dt + g(x)\right\} = 0, \tag{36.23}$$

where

$$g(x) = (2A + B)\,x + \frac{(\lambda+1)A}{x} - \frac{B}{\lambda}\frac{1}{x^3}$$

and $K(x, t)$ is defined by formula (36.17).

To calculate the singular part of the kernel in (36.23), we make use of formula (34.26), and its regular part will be differentiated under the integral sign. Elementary calculations based on the foregoing considerations reduce (36.23) to the final form

$$\frac{1}{2\pi}\int_L \frac{p(t)\, dt}{t-x} + \frac{1}{2\pi}\int_L k_0(x, t)\, p(t)\, dt = f(x), \tag{36.23$'$}$$

$$k_0(x, t) = \frac{1}{2\lambda}\left\{\left(\lambda^2 + 1 + \frac{2}{x^2}\right)\frac{1}{x(xt-1)} - \left[\frac{t^2-1}{t^2}\left(2 + \frac{1}{x^2}\right) + \frac{x^2-1}{x^2}\right] \times \right.$$
$$\left. \times \frac{t}{(xt-1)^2} + \frac{2(t^2-1)(x^2-1)}{x(xt-1)^3} + \frac{\lambda}{x^2 t}\right\}, \tag{36.24}$$

$$f(x) = -\frac{1}{4}g'(x) = -\frac{P}{4}\left(1 + \frac{\lambda+1}{4}\frac{1}{x^2} + \frac{3}{2\lambda}\frac{1}{x^4}\right). \tag{36.25}$$

To Eq. (36.23′) must be added the obvious condition*

$$\int_L p(t)\,dt = 0 \tag{36.26}$$

which expresses the fact that $v^{\pm}(a+0) = v^{\pm}(b-0) = 0$.

By a linear transformation of the variables

$$x = l(\xi+1)+h+1, \qquad t = l(\eta+1)+h+1 \tag{36.27}$$

which maps the cut L with ends $a = 1+h$, $b = 1+h+2l$ into the segment $(-1, 1)$ of the real axis, the singular equation is reduced to the standard form

$$\left.\begin{aligned} &\frac{1}{2\pi}\int_{-1}^{1}\frac{\tau(\eta)\,d\eta}{\eta-\xi}+\frac{1}{2\pi}\int_{-1}^{1}k(\xi,\eta)\,\tau(\eta)\,d\eta = f(\xi),\\ &\int_{-1}^{1}\tau(\eta)\,d\eta = 0 \qquad (-1<\xi<1). \end{aligned}\right\} \tag{36.28}$$

Here

$$\tau(\xi) = p(x), \qquad k(\xi,\eta) = lk_0(x,t), \qquad f(\xi) = f(x). \tag{36.29}$$

It may now be recalled that for $h>0$ the normal stresses σ_x and σ_y become infinite of order 1/2 in the neighbourhood of both ends of the cut. More precisely, in view of the symmetry of the stress field about the x axis, the following relations are valid for points of this axis lying outside the interval $[a, b]$ and close to either of its ends, c (see, for example, N. I. Muskhelishvili [93], p. 610):

$$\sigma_x = \sigma_y = \frac{K}{\sqrt{2|x-c|}}+O(1), \qquad \tau_{xy} = 0, \tag{36.30}$$

where K is the stress intensity factor at the given end.** For points x close to either of the ends c, and situated on the line of the cut we have

$$\begin{aligned} &v^{\pm} = \frac{\varkappa+1}{4\mu}K\sqrt{2|x-c|}+O(|x-c|^{3/2}), \\ &\frac{dv^{+}}{dx} = \pm\frac{\varkappa+1}{4\mu}\frac{K}{\sqrt{2|x-c|}}+O\left(\sqrt{|x-c|}\right); \end{aligned} \tag{36.31}$$

in the second equality the upper sign is taken when $c = a$, and the lower sign, when, $c = b$.

On comparing this equality with (36.21), we find

$$2\sqrt{2|x-c|}\,p(x) = \pm K+O(|x-c|).$$

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* The case $\lambda = 1$, $h = 0$ (h is the distance of the left end of the cut from the boundary of the medium) is excluded.

** The stress intensity factor is often defined as the number $K/\sqrt{2}$ or else $\sqrt{\pi}K$.

From this, with (36.27) and the notation $\tau(\xi) = p(x)$, we obtain

$$K = \pm 2\sqrt{2l} \lim_{\xi \to \mp 1} \sqrt{1 \pm \xi}\, \tau(\xi). \tag{36.32}$$

The plus sign is taken for $\xi = -1$, and the minus sign for $\xi = 1$.

It is clear that when $h > 0$ the number K is, in general, different from zero for both ends, irrespective of the type of boundary condition prescribed along the contour of the hole. When $h = 0$, two cases must be distinguished.

1°. **The hole is filled with a rigid core.** In this case, when the crack ends at the surface of the rigid inclusion, the order of the stress singularity at the end $x = a$, according to the investigations of M. L. Williams [147, 148], depends on Poisson's ratio ν of the material, being approximately equal to 1/3 for $\nu = 0.3$. The stresses near the crack tip still remain infinite, but the stress intensity factor in the sense of (36.30) is zero.

2°. **The hole is not filled with anything**; that is, the crack ends at the surface of a circular cavity free from external forces. Then the tip $x = a$ need not, of course, be considered; the function $p(x)$ is bounded in the neighbourhood of $x = a$ and has a singularity of order 1/2 at the other end.

It appears from the above that in all cases, except when $\lambda = 1$, $h = 0$, the solution of Eq. (36.28) is representable in the form

$$\tau(\xi) = \frac{\tau_0(\xi)}{\sqrt{1-\xi^2}} \tag{36.33}$$

with a continuous function τ_0 on the interval, and it may be sought by the method of Par. 1° in Sec. 13 used in the preceding sections.

The method of solution leads to a system of the form of (34.41)

$$\left.\begin{aligned} &\sum_{\nu=1}^{n} \alpha_{m\nu}\tau_\nu^0 = f_m \qquad (m = 1, 2, \ldots, n), \\ &\sum_{\nu=1}^{n} \tau_\nu^0 = 0, \end{aligned}\right\} \tag{36.34}$$

where the coefficients $\alpha_{m\nu}$ are given by formulas (34.42); the discrete values of the functions $k(\xi, \eta)$ and $f(\xi)$ occurring in these formulas are to be found, of course, from (36.29), (36.24) and (36.25).

Formula (36.32), with representation (36.33) taken into account, becomes

$$K_a = 2\sqrt{l}\,\tau_0(\xi) \quad \text{for } \xi = -1, \qquad K_b = -2\sqrt{l}\,\tau_0(\xi) \quad \text{for } \xi = 1.$$

Noting that

$$\tau_0(\xi) \cong \frac{1}{n} \sum_{k=1}^{n} (-1)^{k+1} \tau_0(\xi_k) \frac{\cos n\vartheta \sin \vartheta_k}{\cos \vartheta - \cos \vartheta_k},$$

where $\tau_0(\xi_k) = \tau_k^0$ is the solution of (36.34), and calculating the values of the right-hand side at the extremities of the interval, we find the stress intensity factors at the ends of the cut as follows:

$$K_a \cong \sqrt{l}\,\frac{2}{n} \sum_{k=1}^{n} (-1)^{k+n} \tau_k^0 \tan\frac{\vartheta_k}{2},$$

$$K_b \cong \sqrt{l}\,\frac{2}{n} \sum_{k=1}^{n} (-1)^{k} \tau_k^0 \cot\frac{\vartheta_k}{2}.$$

The calculations were carried out for the case of a rigid inclusion in a medium which was in a state of plane strain. The values of the ratio

$$\delta = \frac{K}{K_0} \qquad \left(K_0 = \sqrt{l}\,P\right) \tag{36.35}$$

were determined as a function of the parameters of the problem for $\nu = 0.3$. It will be recalled that the number K_0 defines the intensity of infinite stresses in a medium with no inclusion (the stress intensity factor at the ends of a Griffith crack). The δ values are given in the following table.

Stress Intensity Factors Near Rigid Inclusion

h	1.0	0.6	0.4	0.2	0.1	0.06	0.04	0.02	0.01
δ_a	0.927	0.858	0.781	0.470	0.427	0.320	0.248	0.200	0.016
δ_b	0.971	0.955	0.941	0.901	0.897	0.888	0.883	0.881	0.878
l	1	2	3	4	5	6	7	8	9
δ_b	0.878	0.939	0.959	0.969	0.975	0.980	0.983	0.985	0.987

The first two lines of the table give the values of the δ ratio at the isolated ends of the cut when $l = 1$, and the third line gives the values of the same ratio at one end of the cut when the other is at the surface of the inclusion.

As might be expected, the rigid inclusion decreases the stresses at both ends of the cut, its effect being very significant for small h.

The decrease at the right end for a given l becomes greatest when the left end is against the rigid core.

The system of equations (36.34) has been solved for various values of the order n up to $n = 60$. It is of interest to note that the δ values are essentially constant beginning with $n = 20$.

Finally, it is to be remarked that in the case of $\lambda = 1$, $h = 0$ excluded above the potentials (36.15) are no longer of any use at all as they fail to give single-valued displacements. On the basis of (36.15) and (36.21) it can easily be verified that on passing once round the circumference $|z| = R$, $R > 1 + 2l$, in the positive sense the vertical displacement v increases by an amount $\lim\limits_{x \to 1} [v^+(x) - v^-(x)]$ equal to the width of a gap at $x = 1$ resulting from the deformation. This case will be considered separately in the next section.

37. A Crack at a Circular Cavity Surface*

Following H. F. Bueckner, the influence functions are sought in the form (see, for example, H. F. Bueckner and I. Giaever [13])

$$\left.\begin{aligned} \varphi_*(z;\ t) &= -\frac{q(t)}{z-t} + \varphi_*^0(z;\ t), \\ \omega_*(z;\ t) &= -q(t)\left[\frac{1}{z-t} + \left(z - \frac{1}{z}\right)\frac{1}{(z-t)^2}\right] + \omega_*^0(z;\ t), \end{aligned}\right\} \qquad (37.1)$$

where φ_*^0, ω_*^0 are holomorphic functions outside the unit circle which vanish at infinity. As in the preceding section, the regular parts of representation (37.1) can easily be determined by the use of formulas (36.10). For these we have**

$$\left.\begin{aligned} \varphi_*^0(z;\ t) &= -q(t)\left[\frac{z-t}{t^2(zt-1)^2} + \frac{2}{t^3(zt-1)}\right], \\ \omega_*^0(z;\ t) &= -q(t)\left[\frac{1}{t(zt-1)} + \frac{1}{zt^2}\right]. \end{aligned}\right\} \qquad (37.2)$$

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* The solution of the problem was first indicated by L. A. Wigglesworth [146] with the application of Mellin transforms. The more complicated problem of a rotating circular ring with a radial cut at the inner boundary was treated by the influence function and integral equation method in the work of H. F. Bueckner and I. Giaever [13]. The authors discussed in detail the case when the outer circumference of the ring was taken at infinity. In this case, as shown in the above-cited work, the method is fully effective and brings the solution of the problem to an end.

** The authors of the just mentioned work did not use formulas (36.10), giving solutions of the plane problem in closed form, which somewhat complicated the procedure of finding functions (37.2).

On the basis of (37.1) and (37.2) the required potentials of the problem are represented as

$$\left.\begin{aligned}\varphi_*(z) &= -\frac{1}{\pi}\int_L K_1(z,\ t)\,q(t)\,dt,\\ \omega_*(z) &= -\frac{1}{\pi}\int_L K_2(z,\ t)\,q(t)\,dt,\end{aligned}\right\}\qquad (37.3)$$

$$K_1(z,\ t)=\frac{1}{z-t}+\frac{z-t}{t^2(zt-1)^2}+\frac{2}{t^3(zt-1)},$$

$$K_2(z,\ t)=\frac{1}{z-t}+\left(z-\frac{1}{z}\right)\frac{1}{(z-t)^2}+\frac{1}{t(zt-1)}+\frac{1}{zt^2}.$$

To potentials (37.3) must be added the functions $\varphi_0(z)$, $\psi_0(z)$ given by formulas (36.11) with $\lambda = 1$.

The mechanical significance of the density function $q(x)$ in (37.3) is defined similarly to the case of $h > 0$ by the relation

$$\pm q_*'(x)=\frac{2\mu}{\varkappa+1}v^{\pm}\quad\text{on}\quad L. \qquad (37.4)$$

It is easily verified that representation (37.3) can be obtained from (36.15) by integration by parts if the condition

$$\int_L p(x)\,dx=0$$

is fulfilled, and conversely formulas (36.15) follow from (37.3) subject to the additional conditions

$$q(a) = q(b) = 0.$$

The condition for zero external forces along the sides of the cut, which is the only one to be satisfied when there are available (37.3), leads, as before, to an equality of the form of (36.23). This equality is written as

$$\frac{1}{\pi}\int_L K(x,\ t)\,q(t)\,dt=g(x)+c,$$

where c is an arbitrary real constant. In our case it becomes

$$\frac{1}{2\pi}\int_L\frac{q(t)\,dt}{t-x}+\frac{1}{2\pi}\int_L k_0(x,\ t)\,q(t)\,dt=f_0(x)+c, \qquad (37.5)$$

where

$$-k_0(x,\ t)=\frac{1}{t^3(xt-1)}+\frac{(t-x)(x^2-1)}{xt(xt-1)^3}+\frac{1}{xt^2},$$

$$f_0(x)=-\frac{p}{8}\left(2x-\frac{1}{x}-\frac{1}{x^3}\right).$$

There is no additional condition of the form of (36.26) in our case. This time, in accordance with (37.4), it is necessary to have a solution of (37.5) bounded at both ends of the cut. On the other hand, as pointed out in Sec. 13, singular equations do not in general admit such solutions. To ensure a bounded solution for (37.5), we must make use of the constant c, as yet arbitrary, appearing on the right-hand side of the equation.

The standard form of Eq. (37.5) is the following:

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{\tau_0(\eta)\,d\eta}{\eta-\xi}+\frac{1}{2\pi}\int_{-1}^{1}k(\xi,\ \eta)\,\tau_0(\eta)\,d\eta=f(\xi)+c, \tag{37.6}$$

$$\tau_0(\xi)=q(x),\qquad k(\xi,\ \eta)=lk_0(x,\ t),\qquad f(\xi)=f_0(x),$$
$$x=l(\xi+1)+1,\qquad t=l(\eta+1)+1,$$

where l is the half-length of the cut. The kernel $k(\xi,\ \eta)$ is continuous in the closed square $-1\leqslant\xi,\ \eta\leqslant 1$ from which the vertex at the lower left corner is removed.

According to (37.4), the solution of Eq. (37.6) must become zero at $\xi=1$ and remain bounded near the other end; it cannot vanish at $\xi=-1$.

Hence, if we introduce a new unknown function

$$\tau(\xi)=\sqrt{1+\xi}\,\tau_0(\xi)\qquad(\tau_0(\xi)=q(x)), \tag{37.7}$$

the method of solution developed in Par. 3° of Sec. 13 will be equally applicable to the integral equation in τ. The transformed equation is of the form

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{\tau(\eta)\,d\eta}{\eta-\xi}+\frac{1}{2\pi}\int_{-1}^{1}k_*(\xi,\ \eta)\,\tau(\eta)\,d\eta=f_*(\xi)+\sqrt{1+\xi}\,c, \tag{37.8}$$

where

$$k_*(\xi,\ \eta)=\frac{1}{\sqrt{1+\eta}}\left[\sqrt{1+\xi}\,k(\xi,\ \eta)-\frac{1}{\sqrt{1+\eta}+\sqrt{1+\xi}}\right],$$
$$f_*(\xi)=\sqrt{1+\xi}\,f(\xi).$$

In passing to the approximate solution (13.8), we note that

$$\tau(\xi)=\sqrt{1-\xi^2}\,\tau^*(\xi), \tag{37.9}$$

where τ^* is a continuous function on the interval, and replace the unknown $\tau(\xi)$ by the Lagrange interpolation polynomial ($\xi=$

$= \cos \vartheta)$

$$\tau(\xi) \cong \frac{1}{n+1} \sum_{k=1}^{n} \tau(\xi_k) \frac{(-1)^{k+1} \sin \vartheta_k \sin (n+1) \vartheta}{\cos \vartheta - \cos \vartheta_k}, \qquad (37.10)$$

$$\xi_k = \cos \vartheta_k, \quad \vartheta_k = \frac{k\pi}{n+1} \qquad (k = 1, 2, \ldots, n).$$

It may be recalled that the right-hand side of (37.10) is of the form of (37.9), with the corresponding function τ^* representing a polynomial in ξ of degree not higher than $n - 1$.

For the determination of the values of τ_k and the constant c, we obtain a system of equations

$$\sum_{\nu=1}^{n} \alpha_{m\nu}\tau_\nu - \sqrt{1+\xi_m}\, c = f_*(\xi_m), \qquad (m = 1, 2, \ldots, n). \quad (37.11)$$

Here

$$\alpha_{m\nu} = \frac{\sin \vartheta_\nu}{n+1} \left[\frac{\varepsilon_{m\nu}}{\cos \vartheta_\nu - \cos \vartheta_m} + \frac{1}{2} k_*(\xi_m, \eta_\nu) \right] =$$

$$= \frac{\sin \frac{\vartheta_\nu}{2}}{2(n+1)} \left[\frac{4\varepsilon_{m\nu} \cos \frac{\vartheta_\nu}{2}}{\cos \vartheta_\nu - \cos \vartheta_m} - \frac{1}{\cos \frac{\vartheta_\nu}{2} + \cos \frac{\vartheta_m}{2}} + 2\cos\frac{\vartheta_m}{2}\, k(\xi_m, \eta_\nu) \right],$$

$$\varepsilon_{m\nu} = \begin{cases} 1, & |m-\nu| = 1, 3, \ldots, \\ 0, & |m-\nu| = 0, 2, \ldots . \end{cases}$$

To obtain the lacking equation in (37.11), we make use of condition (13.23) which guarantees the existence of a bounded solution at both ends. This condition in our case takes the form

$$\int_{-1}^{1} \frac{f_*(\xi) + \sqrt{1+\xi}\, c - R[\tau; \xi]}{\sqrt{1-\xi^2}}\, d\xi = 0, \qquad (37.12)$$

$$R[\tau; \xi] = \frac{1}{2\pi} \int_{-1}^{1} k_*(\xi, \eta)\, \tau(\eta)\, d\eta. \qquad (37.13)$$

Equation (37.12) will be taken in discrete form and for that it is necessary to apply the mechanical quadrature formulas to the two preceding integrals. To evaluate the left-hand side of (37.12), it is advisable to use formula (13.8) repeatedly employed above, and the right-hand side of (37.13) is most conveniently replaced by a sum according to another quadrature formula of the same type:

$$\int_{-1}^{1} \sqrt{1-\xi^2}\, F(\xi)\, d\xi = \frac{\pi}{n+1} \sum_{k=1}^{n} \sin^2 \vartheta_\nu F(\cos \vartheta_\nu). \qquad (37.14)$$

We finally come to the linear equation

$$\sum_{\nu=1}^{n} \alpha_{n+1,\,\nu}\tau_\nu - \frac{2\sqrt{2}}{\pi}c = \frac{1}{n}\sum_{s=1}^{n} f_*(\xi_s'), \tag{37.15}$$

where

$$\alpha_{n+1,\,\nu} = \frac{\sin\frac{\vartheta_\nu}{2}}{2n(n+1)}\sum_{s=1}^{n}\left[2\cos\frac{\vartheta_s'}{2}k(\xi_s',\ \eta_\nu) - \frac{1}{\cos\frac{\vartheta_\nu}{2}+\cos\frac{\vartheta_s'}{2}}\right],$$

$$\xi_s' = \cos\vartheta_s', \quad \vartheta_s' = \frac{2s-1}{2n}\pi, \qquad (s = 1,\ 2,\ \ldots,\ n).$$

Equalities (37.11) and (37.15) determine a complete system of linear algebraic equations.

We proceed to the calculation of the characteristic quantities of the problem. When the solution $\tau(\xi)$ of Eq. (37.8) has been determined, the stress intensity factor at the end of the cut ($x = b$) is easily found from formulas (36.31), (37.4) and (37.7). As elementary calculations show,

$$K = \frac{1}{\sqrt{l}}\lim_{\xi\to 1}\frac{\tau(\xi)}{\sqrt{1-\xi}} = \frac{1}{\sqrt{2l}}\lim_{\vartheta\to 0}\frac{\tau(\cos\vartheta)}{\sin\frac{\vartheta}{2}}.$$

Substituting the right-hand side of (37.10) and passing to the limit as $\vartheta \to 0$, we find an approximate value of the factor K as

$$K = \sqrt{\frac{2}{l}}\sum_{k=1}^{n}(-1)^{k+1}\tau_k\cot\frac{\vartheta_k}{2}. \tag{37.16}$$

Determine next the maximum value of the normal stress σ_ϑ on the contour of the hole

$$\max\sigma_\vartheta = 4\,\mathrm{Re}\,\{\varphi_*'(z)\} + 3P \qquad (z = -1). \tag{37.17}$$

From (37.3), by differentiation under the integral we determine the first derivative of the potential

$$\varphi_*'(z) = \frac{1}{\pi}\int_L\left[\frac{1}{(z-t)^2} + \frac{1}{t^2(zt-1)^2} + \frac{2(z-t)}{t(zt-1)^3}\right]q(t)\,dt.$$

On putting $z = -1$ and substituting the value of φ_*' in (37.17), we obtain

$$\sigma_\vartheta\big|_{z=-1} = \frac{4}{\pi}\int_L\frac{q(t)}{t^2}dt + 3P. \tag{37.18}$$

Taking into account (37.7) and using the quadrature formula (37.14), we determine an approximate value of the maximum contour stress as

$$\sigma_\vartheta\big|_{z=-1} = \frac{4l}{\pi}\int_{-1}^{1}\frac{\tau(\xi)\,d\xi}{\sqrt{1+\xi}\,[l(1+\xi)+1]^2} + 3P =$$

$$= \frac{8l}{\sqrt{2}\,(n+1)}\sum_{k=1}^{n}\frac{\tau_k\sin\dfrac{\vartheta_k}{2}}{[l(1+\cos\vartheta_k)+1]^2} + 3P. \qquad (37.19)$$

System (37.11), (37.15) has been solved with $P = 1$ using an electronic computer and its solution τ_k has been employed to determine the values of ratio (36.35) and the maximum stress σ_ϑ as a function of the length of the cut. The numerical results for δ are given below.

l	10.0	5.0	1.0	0.2	0.04	0.01
δ	1.1338	1.2006	1.6281	2.9460	4.3970	4.9063

The ratio δ increases with decreasing length l, and as the latter is increased it diminishes and tends to a limit equal to unity. The maximum contour stress σ_ϑ increases steadily with increasing l.

We revert to formula (37.18), after first rewriting it in the form

$$\sigma_\vartheta\big|_{z=-1} = \frac{4}{\pi}\,\frac{2\mu}{\varkappa+1}\int_{1}^{1+2l}\frac{v^+(t,\,0)\,dt}{t^2} + 3P. \qquad (37.20)$$

It should be noted, to begin with, that for a Griffith crack on the segment $[-\frac{1}{2}-l,\ \frac{1}{2}+l]$ of the real axis when the edges of the crack are free from external forces and the medium is extended at infinity by forces P perpendicular to the line of the cut we have

$$\varphi(z) = \frac{P}{2}\sqrt{z^2-c^2} - \frac{P}{4}z + \text{constant}, \qquad c = l + \frac{1}{2}.$$

From this and on the basis of the Kolosov-Muskhelishvili formula

$$2\mu\,(u+iv) = (\varkappa+1)\,\varphi(z) - \Omega(z,\bar{z}),$$

$$\Omega(z,\bar{z}) = \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)},$$

for points x situated on the line of the Griffith crack, we obtain

$$\frac{2\mu}{\varkappa+1}v^+(x,\,0) = \frac{P}{2}\sqrt{c^2-x^2} + \text{constant}.$$

After the origin is transferred to the point $x = -c$, we have

$$\frac{2\mu}{\varkappa+1} v^{+}(t-c,\ 0)=\frac{P}{2}\sqrt{2ct-t^2}+\text{constant} \qquad (0\leqslant t\leqslant 2l+1). \tag{37.21}$$

The effect of a circular cavity on the stress field near it is, of course, insignificant. At an appreciable distance from the origin, the leading term in the expression for $\frac{2\mu}{\varkappa+1}v^{+}(t,\ 0)$ under the integral in (37.20) will not differ from the leading term on the right-hand side of the preceding equality, i.e., from $\frac{P}{2}\sqrt{ct}$. Hence, on the basis of (37.20) and (37.21) the following asymptotic formula is valid for large l:*

$$\sigma_{\vartheta}\big|_{z=-1}=K_{\infty}\sqrt{l}\,[1+o(1)], \qquad K_{\infty}=\text{constant}.$$

The constant K_{∞}, which may naturally be defined as the stress intensity factor at the end of a semi-infinite cut, is calculated by the approximate formula

$$K_{\infty}=\frac{4\sqrt{2l}}{n+1}\sum_{k=1}^{n}\frac{\tau_k \sin\frac{\vartheta_k}{2}}{[l+(1+\cos\vartheta_k)+1]^2} \qquad (\text{for large } l).$$

37a. Notched Half-plane

A straight crack of length $b = 2l$ ends at the boundary of an elastic half-plane at right angles, the faces of the cut and the edge of the half-plane are free from external forces and the medium is subjected at infinity to tensile forces P perpendicular to the line of the cut. The region S occupied by the elastic medium is placed in the upper half-plane, as shown in Fig. 7, and the real axis is denoted by L.

In the notation

$$\omega(z) = z\varphi'(z) + \psi(z),$$

$$\Omega(z,\ \bar{z}) = \varphi(z) + \overline{z\varphi'(z)} + \overline{\psi(z)} = \varphi(z) + \overline{\omega(z)} + (z-\bar{z})\,\overline{\varphi'(z)}$$

the boundary conditions are written as

$$\varphi(t) + \overline{\omega(t)} = \text{constant on } L, \tag{37a.1}$$

$$\Omega^{\pm}(z,\ \bar{z}) = \text{constant} \qquad (z = iy, \quad 0 < y < b). \tag{37a.2}$$

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* For this statement I am indebted to R. L. Salganik.

Since the stress field is uniform at infinity,

$$\left.\begin{aligned} \varphi(z) &= \varphi_*(z) + \frac{P}{4} z, \\ \omega(z) &= \omega_*(z) - \frac{P}{4} z, \end{aligned}\right\} \tag{37a.3}$$

where φ_*, ω_* are holomorphic functions in S which admit the following asymptotic representation for large $|z|$:

$$\varphi_*(z) = o(1), \qquad \omega_*(z) = o(1) + \text{constant}. \tag{37a.4}$$

It is advantageous in our case to seek the influence functions in the form

$$\left.\begin{aligned} \varphi_*(z;\ \eta) &= -\frac{q(\eta)}{z - i\eta} + \varphi_*^0(z;\ \eta), \\ \omega_*(z;\ \eta) &= q(\eta)\left[\frac{1}{z - i\eta} + \frac{2z}{(z - i\eta)^2}\right] + \\ &+ \omega_*^0(z;\ \eta) \qquad (0 \leqslant \eta \leqslant b), \end{aligned}\right\} \tag{37a.5}$$

where $q(\eta)$ is some real function on the interval $[0, b]$ and φ_*^0, ω_*^0 are holomorphic functions in the region S which are subject to conditions (37a.4).

For the determination of the functions φ_*^0, ω_*^0 we have condition (37a.1) which gives

$$\varphi_*^0(t;\ \eta) + \overline{\omega_*^0(t;\ \eta)} = f_*(t;\ \eta), \tag{37a.6}$$

$$f_*(t;\ \eta) = q(\eta)\left[\frac{1}{t - i\eta} - \frac{1}{t + i\eta} - \frac{2t}{(t + i\eta)^2}\right].$$

It is well known that if a given function $f(t)$ on L represents the boundary values of some function $F(z)$ holomorphic in the upper half-plane, then the function $\overline{f(t)}$ represents the boundary values on L of the function $\overline{F}(\overline{z})$ holomorphic in the lower half-plane. Hence, the integral formula and theorem of Cauchy for a half-plane (see, for example, N. I. Muskhelishvili [93], Sec. 72), with (37a.4), will immediately give a solution of problem (37a.6) in the form

$$\varphi_*^0(z;\ \eta) = \frac{1}{2\pi i}\int_L \frac{f_*(t;\ \eta)\,dt}{t - z} + \text{constant},$$

$$\omega_*^0(z;\ \eta) = \frac{1}{2\pi i}\int_L \frac{\overline{f_*(t;\ \eta)}\,dt}{t - z} + \text{constant}.$$

On the same basis (formulas and theorems of Cauchy) the preceding integrals are evaluated in an elementary way. Inserting the expressions for them in (37a.5), we find, rejecting the non-essential constants,

$$\left.\begin{aligned} &\varphi_*(z;\ \eta)=K_1(z,\ \eta)\,q(\eta),\\ &\omega_*(z;\ \eta)=K_2(z,\ \eta)\,q(\eta);\\ -&K_1(z,\ \eta)=\frac{1}{z-i\eta}+\frac{1}{z+i\eta}+\frac{2z}{(z+i\eta)^2},\\ &K_2,(z,\ \eta)=-K_1(z,\ -\eta). \end{aligned}\right\} \qquad (37\text{a}.7)$$

Assume now

$$\left.\begin{aligned} \varphi_*'(z)&=\frac{1}{\pi}\int\limits_0^b K_1(z,\ \eta)\,q(\eta)\,d\eta,\\ \omega_*(z)&=\frac{1}{\pi}\int\limits_0^b K_2(z,\ \eta)\,q(\eta)\,d\eta. \end{aligned}\right\} \qquad (37\text{a}.8)$$

According to condition (37a.2), the expression $\Omega(z,\ \bar{z})$ maintains a constant value along the cut. Hence, on the basis of formula (36.4) the following equality holds for horizontal elastic displacements of the points on the sides of the cut:

$$2\mu u^{\pm}=(\varkappa+1)\quad \text{e}\ \varphi_*^{\pm}(z) \qquad (z=iy,\ 0<y<b).$$

From this and from representation (37a.8), by the Sokhotskii-Plemelj formula there results

$$\pm q(y)=\frac{2\mu}{\varkappa+1}u^{\pm}(0,\ y) \qquad (0<y<b). \qquad (37\text{a}.9)$$

From (37a.3) and (37a.8) it is easily seen that the expression $\Omega(z,\ \bar{z})$ takes pure imaginary values everywhere on the imaginary axis (the ends of the cut are excluded). Elementary calculations based on formulas (37a.7) show that on this axis the kernel of the representation for $\Omega(z,\ \bar{z})$:

$$K(z,\ \eta)=K_1(z,\ \eta)+\overline{K_2(z,\ \eta)}+(z-\bar{z})\,\frac{\overline{\partial K_1}}{\partial z}$$

is of the form

$$K(iy,\ \eta)=-\frac{2}{i}\left[\frac{1}{y-\eta}+\frac{1}{y+\eta}+\frac{2y}{(y+\eta)^2}-\frac{4y}{(y+\eta)^3}\right]$$

and the boundary condition (37a.2) leads to the integral equation

$$\frac{1}{2\pi}\int\limits_0^b\frac{q(\eta)\,d\eta}{\eta-y}+\frac{1}{2\pi}\int\limits_0^b k_0(y,\ \eta)\,q(\eta)\,d\eta=\frac{P}{4}y+c \qquad (0<y<b). \qquad (37\text{a}.10)$$

Here c is some as yet arbitrary real constant and

$$-k_0(y,\ \eta)=\frac{1}{(\eta+y)}+\frac{2y(\eta-y)}{(\eta+y)^3}.$$

After transformation of the variables

$$y = l(1+\xi), \qquad \eta = l(1+t),$$

where l is the half-length of the cut, this equation becomes

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{p(t)\,dt}{t-\xi}+\frac{1}{2\pi}\int_{-1}^{1}k(\xi,\ t)\,p(t)\,dt = f(\xi)+c \qquad (-1<\xi<1), \tag{37a10'}$$

$$-k(\xi,\ t)=\frac{1}{2+t+\xi}+\frac{2(t-\xi)(1+\xi)}{(2+t+\xi)^3}, \tag{37a.11}$$

$$p(\xi)=q(y), \qquad f(\xi)=\frac{P}{4}\,l\,(1+\xi).$$

Equation (37a.10′) is not different from Eq. (37.6) in either its structure or the meaning of the conditions imposed on the solution, and hence the method of approximate solution developed in the preceding section, following equality (37.6), is equally applicable to it.

For the determination of discrete values of the function

$$\tau(\xi)=\sqrt{1+\xi}\,p(\xi),$$

vanishing at both ends of the cut, and the constant c we have the following system of $n+1$ equations:

$$\left.\begin{aligned}&\sum_{\nu=1}^{n}\alpha_{m\nu}\tau_\nu-\sqrt{1+\xi_m}\,c=\frac{Pl}{4}(1+\xi_m)^{3/2}\\ &\qquad\qquad\qquad\qquad (m=1,\ 2,\ \ldots,\ n),\\ &\sum_{\nu=1}^{n}\alpha_{n+1,\,\nu}\tau_\nu-\frac{2\sqrt{2}}{\pi}\,c=\frac{Pl}{4n}\sum_{s=1}^{n}(1+\xi_s')^{3/2},\end{aligned}\right\} \tag{37a.12}$$

$$\alpha_{m\nu}=\frac{\sin\frac{\vartheta_\nu}{2}}{2(n+1)}\left[\frac{4\varepsilon_{m\nu}\cos\frac{\vartheta_\nu}{2}}{\cos\vartheta_\nu-\cos\vartheta_m}+2\cos\frac{\vartheta_m}{2}\,k(\xi_m,\ t_\nu)-\frac{1}{\cos\frac{\vartheta_\nu}{2}+\cos\frac{\vartheta_m}{2}}\right],$$

$$\alpha_{n+1,\,\nu}=\frac{\sin\frac{\vartheta_\nu}{2}}{2n(n+1)}\left[\sum_{s=1}^{n}2\cos\frac{\vartheta_s'}{2}\,k(\xi_s',\ t_\nu)-\frac{1}{\cos\frac{\vartheta_\nu}{2}+\cos\frac{\vartheta_s'}{2}}\right],$$

$$\xi_m=t_m=\cos\vartheta_m, \quad \vartheta_m=\frac{m\pi}{n+1}, \quad \xi_m'=\cos\vartheta_m', \ \vartheta_m'=\frac{2m-1}{2n}\pi.$$

The values of $k(\xi_m,\ t_\nu)$ are determined from (37a.11).

Equation (37a.10) written in terms of the non-dimensional variables contains the first power of the half-length of the cut l on the right-hand side. Hence, the ratio

$$\delta = \frac{K}{K_0} \qquad (K_0 = \sqrt{l}\, P), \tag{37a.13}$$

where K is the stress intensity factor at the end of the cut, will be independent of the dimension l in our case. It is calculated in terms of the solution of system (37a.12) by the formula

$$\delta = \sqrt{2} \sum_{k=1}^{n} (-1)^k \tau_k \cot \frac{\vartheta_k}{2}. \tag{37a.14}$$

In conclusion it should be noted that the typical example of a crack at a free surface boundary considered here attracted at one time the attention of many prominent scientists who developed a number of approximate methods for its solution (L. A. Wigglesworth [145], G. R. Irwin [35], H. F. Bueckner [12], W. T. Koiter [60]; see also H. F. Bueckner and I. Giaever [13]). It was of special interest to find the ratio (37a.13), and this was determined to a lesser or greater degree of accuracy. The values of the δ ratio found by the afore-mentioned authors lie between 1.5 and 1.6. According to W. T. Koiter, the value of $\frac{\delta}{\sqrt{2}}$ is 1.1215 with high accuracy.

On solving system (37a.10) with $n = 30$, formula (37a.14) gives $\delta = 1.7$.

38. A Stiffened Plate Containing an Isolated Crack

The problem of the reinforcing of plates becomes particularly complicated in the case when the matrix in the zone of reinforcement has defects in the form of holes, notches, cracks, etc., and the important thing then is to determine the mutual effect of two opposite factors, viz. the reinforcing of the medium and its weakening.

One of the problems of this kind was considered in Sec. 34. Such problems have been previously encountered in the literature, being the subject of study by many authors.*

In the present section, for the case of a straight stringer extending to infinity with both its ends, an approximate method is indicated for the solution of a class of problems which permits their reduction to integral equations with regular kernels and a subsequent application to the solution of a certain analogue of the well-known Schwarz algorithm. The class of problems included into consideration is determined by the requirement that the problem of the equilibrium of a plate weakened by a notch (or a system of notches) but not reinforced with a stiffener should admit a solution in closed form. By way of illustration we restrict our consideration to the case when an infinite plate has a cut of finite length, and an infinite stringer (of uniform section) is continuously attached to the plate along a straight line perpendicular to the line of the cut.*

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* Some references may be found in the paper by R. Greif and J. L. Sanders, Jr. [32] mentioned at the beginning of the present part.

Refer the region occupied by the elastic plate to the plane of the variable $z = x + iy$, place the axis of the stringer on the real axis which will be further denoted by L, and make the cut coincide with a segment l of the imaginary axis. The end points of the cut are designated by ai and bi; $0 < a < b$.

As before, suppose that the stringer has no flexural rigidity. For simplicity, it is assumed that no external forces are applied to the stringer, the stresses and rotation vanish at infinity and the faces of the cut are acted on by given forces.

It should be borne in mind that it is also necessary to distinguish the left and right sides (in relation to the positive direction of the co-ordinate axis) and to assign to the associated quantities the respective plus and minus signs.

1°. Consider first a problem involving no crack. It will be assumed that the plate is subjected to distributed forces X, Y applied directly to the axis of the stringer and that the stresses and rotation vanish at infinity.

The problem with no crack is analogous to that considered in Sec. 31 and is more elementary. We reproduce here the arguments of the above-mentioned section with some necessary modifications in abbreviated form.

The condition for equilibrium of the stringer attached to the plate throughout its length is obtained from formulas (31.1) and (31.1a) by a limiting process as $dx \to 0$ if, in addition, the external forces $X(x)$ and $Y(x)$ are incorporated in these equalities. We have

$$h\,[\tau_{xy}^{+} - \tau_{xy}^{-} + i\,(\sigma_y^{+} - \sigma_y^{-})] + X(x) + iY(x) + \frac{d}{dx}N(x) = 0 \text{ on } L. \quad (38.1)$$

To the above equality, as in Sec. 31, must be added conditions (31.3), the first of which (in the displacements) being taken in the form

$$u^{+\prime} + iv^{+\prime} = u^{-\prime} + iv^{-\prime} \text{ on } L, \quad (38.2)$$

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* The method of solution proposed below is presented in the author's paper [51]. An alternative method for the approximate solution of this problem, when a plate is under simple tension in the direction of the stringer axis, is contained in the work of two authors to which reference is made in the preceding footnote. We further mention a paper by J. M. Bloom [10] dealing with the case of a (straight and infinite) riveted stringer in a sheet with a circular hole.

where a prime denotes differentiation with respect to x. In using conditions (38.2) we imply in what follows the validity of the equalities

$$\left(\frac{\partial u}{\partial x}\right)^{\pm} + i\left(\frac{\partial v}{\partial x}\right)^{\pm} = u^{\pm\prime} + iv^{\pm\prime}. \tag{38.3}$$

Condition (31.3) in the extensions, namely

$$\left(\frac{\partial u}{\partial x}\right)^{+} = \varepsilon^0,$$

on the basis of the previously mentioned formula $N = E_0S_0\varepsilon^0$ gives, with formula (38.3),

$$\frac{d}{dx} N(x) = E_0S_0 \frac{du^{2+}}{dx^2}. \tag{38.4}$$

Consequently, the condition for equilibrium of the stringer (38.1) becomes, finally,

$$h\left[\tau_{xy}^{+} - \tau_{xy}^{-} + i(\sigma_y^{+} - \sigma_y^{-})\right] + X(x) + iY(x) + E_0S_0 \frac{d^2u^+}{dx^2} = 0 \text{ on } L. \tag{38.5}$$

Relations (38.2) and (38.5) constitute the boundary conditions of our problem.*

The well-known complex representations of displacement and stress components which have been repeatedly employed above reduce the problem (38.2), (38.5) to the determination of sectionally holomorphic functions $\Phi(z)$, $\Psi(z)$ having the whole real axis L as a line of discontinuity and satisfying on it the boundary conditions**

$$\begin{aligned}\Phi^+(t) + \overline{\Phi^+(t)} + t\overline{\Phi'^+(t)} + \overline{\Psi^+(t)} = \Phi^-(t) + \overline{\Phi^-(t)} + {}\\ + t\overline{\Phi'^-(t)} + \overline{\Psi^-(t)} + iK_0 \operatorname{Re}\left\{\frac{d}{dt}\left[\varkappa\Phi^+(t) - \overline{\Phi^+(t)} - {}\right.\right.\\ \left.\left. - t\overline{\Phi'^+(t)} - \overline{\Psi^+(t)}\right]\right\} + f_0(t),\end{aligned} \tag{38.6}$$

$$\begin{aligned}\varkappa\Phi^+(t) - \overline{\Phi^+(t)} - t\overline{\Phi'^+(t)} - \overline{\Psi^+(t)} = \varkappa\Phi^-(t) - \overline{\Phi^-(t)} - {}\\ - t\overline{\Phi'^-(t)} - \overline{\Psi^-(t)} \quad (t \text{ on } L)\end{aligned} \tag{38.7}$$

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* Whenever the region occupied by the elastic medium is infinite, it is much more convenient to proceed from a boundary condition of the form of (38.5) and not from a condition obtained by integrating it, as in Sec. 31. However, we could not have done it in Sec. 31 because the axial load p_0 would have been left out of the consideration.

** Sectional holomorphy (in the plane) here implies that $\Phi(z)$ and $\Psi(z)$ are holomorphic everywhere in S^+ and in S^- and are continuously extendible to all points of L from the left and from the right. The behaviour of these functions at infinity is of no importance for the present.

with

$$\left.\begin{aligned} &\Phi(z)=\varphi'(z), &&\Psi(z)=\psi'(z),\\ &\varkappa=\frac{3-\nu}{1+\nu}, &&K_0=\frac{E_0S_0}{2\mu h},\\ &f_0(t)=\frac{i}{h}\,[X(t)+iY(t)]. && \end{aligned}\right\} \qquad (38.8)$$

We introduce a new sectionally holomorphic function $\chi(z)$ in the z plane by the equality

$$\chi(z)=\Phi(z)+z\Phi'(z)+\Psi(z) \qquad (38.9)$$

and add equalities (38.6) and (38.7). The boundary conditions for the functions Φ, χ are then written in the form

$$\left.\begin{aligned} &(\varkappa+1)\,\Phi^+(t)=(\varkappa+1)\,\Phi^-(t)+\\ &+iK_0\operatorname{Re}\left\{\frac{d}{dt}\left[\varkappa\Phi^+(t)-\overline{\chi^+(t)}\right]\right\}+f_0(t),\\ &\varkappa\Phi^+(t)-\overline{\chi^+(t)}=\varkappa\Phi^-(t)-\overline{\chi^-(t)}. \end{aligned}\right\} \qquad (38.10)$$

The last equality is rewritten as

$$\varkappa\Phi^+(t)+\bar\chi^+(t)=\varkappa\Phi^-(t)+\bar\chi^-(t), \qquad \left(\bar\chi(z)=\overline{\chi(\bar z)}\right). \quad (38.11)$$

The function $\bar\chi(z)$ is obviously also sectionally holomorphic in the entire plane and has the same real axis L for a line of discontinuity.

Of the given functions X, Y we shall require that they be finite and integrable on any finite segment of the straight line and satisfy, for large $|t|$, the condition

$$f_0(t)=o\left(\frac{1}{t}\right). \qquad (38.12)$$

Then, in view of the previous assumption about the absence of the stresses and rotation at infinity it may be taken that the following estimates are valid for large $|z|$ in each of the half-planes S^+ and S^- (cf. N. I. Muskhelishvili [93], Sec. 112)

$$\Phi(z)=O(z^{-1}), \qquad \Phi'(z)=O(z^{-2}), \quad \Psi(z)=O(z^{-1}). \qquad (38.13)$$

From this and the continuity condition (38.11) we now easily deduce the validity of the equality

$$\varkappa\Phi(z)+\bar\chi(z)=0 \qquad \text{in the entire } z \text{ plane} \qquad (38.14)$$

from which, as before, for points of the real axis we derive the relation

$$\overline{\chi'^+(t)}=-\varkappa\,\Phi'^-(t) \quad \text{on } L.$$

The above equality enables us to exclude from the consideration the unknown $\chi(z)$, and according to (38.10) we arrive at the problem of determining the single function $\Phi(z)$, sectionally holomorphic in the plane, from the boundary condition

$$\Phi^+(t) = \Phi^-(t) + i\lambda \operatorname{Re}\{\Phi'^+(t) + \Phi'^-(t)\} + f(t) \text{ on } L, \qquad (38.15)$$

where

$$\lambda = \frac{\varkappa K_0}{\varkappa+1}, \qquad f(t) = \frac{f_0(t)}{\varkappa+1}. \qquad (38.16)$$

Assuming

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{\omega(\tau)\,d\tau}{\tau - z}, \qquad (38.17)$$

where $\omega(t) = \mu(t) + i\nu(t)$ is a new unknown function on the axis, and substituting the limiting values, from the left and right, on L of the function $\Phi(z)$ and its derivative in condition (38.15), we find

$$\omega(t) + \frac{\lambda}{\pi t}\int_L \frac{\nu'(\tau)\,d\tau}{\tau - t} = f(t). \qquad (38.18)$$

Separating here the real and imaginary parts, we have

$$\mu(t) = -\frac{Y(t)}{h(\varkappa+1)}, \qquad (38.19)$$

$$\nu(t) - \frac{\lambda}{\pi}\int_L \frac{\nu'(\tau)\,d\tau}{\tau - t} = g(t) \quad \text{on} \quad L, \qquad (38.20)$$

where

$$g(t) = \frac{X(t)}{h(\varkappa+1)}.$$

The solution of the integro-differential equation (38.20) can easily be constructed in terms of Fourier integrals. By applying the Fourier transformation to both its sides, we obtain

$$N(t) = \frac{1 + \frac{\lambda}{\pi i}\, t \operatorname{sgn} t}{1 + \frac{\lambda^2}{\pi^2} t^2} G(t) \qquad (38.21)$$

with

$$N(t) = \int_L \nu(\tau) e^{it\tau}\,d\tau, \qquad G(t) = \int_L g(\tau) e^{it\tau}\,d\tau.$$

Application of the inverse Fourier transformation to equality (38.21) determines the unknown function $\nu(t)$ in the following final form:

$$\nu(t) = \int_L g(t-x)\,\eta(x)\,dx + \frac{\lambda}{\pi}\int_L g(t-y)\,dy \int_L \frac{\eta'(x)\,dx}{x-y}, \qquad (38.22)$$

where $\eta(t)$ denotes the inverse Fourier transform of the function

$$\frac{\pi^2}{\pi^2 + \lambda^2 t^2},$$

namely,

$$\eta(t)=\frac{\pi}{2}\int_{L}\frac{e^{-it\tau}\,d\tau}{\pi^2+\lambda^2\tau^2}.$$

The above integral is easily evaluated for any real λ, giving

$$\eta(t)=\frac{\pi}{2\,|\lambda|}\,e^{-\pi\left|\frac{t}{\lambda}\right|}. \tag{38.23}$$

It should be borne in mind that for the existence of integrals over the infinite straight line L understood as a Cauchy principal value* the unknown density function $\omega(t)$ must be subjected to certain specific conditions on the same straight line (see N. I. Muskhelishvili [93], Sec. 71). As appears from formulas (38.19) and (38.22) giving an explicit expression for $\omega(t)$, these conditions must be provided by appropriate requirements imposed on the given functions $X(t)$ and $Y(t)$. The requirements necessary for these purposes are considered to be fulfilled.

Ultimately the function $\Phi(z)$, that is, the Cauchy-type integral (38.17), is found explicitly (by quadratures), whereupon the second unknown function $\Psi(z)$ is determined from formulas (38.14) and (38.9). We have for it

$$\Psi(z)=-\varkappa\bar{\Phi}(z)-z\Phi'(z)-\Phi(z). \tag{38.24}$$

Note. For applications it may be of interest to investigate the case when the stresses in a reinforced plate are produced by a concentrated load $(p,\ q)$ applied at an arbitrary point $z_0=x_0+iy_0$ not lying on the x axis $(y_0\neq 0)$.

In this case

$$X(t)=Y(t)=0$$

and the solution of the homogeneous problem (38.10) is represented in the form (N. I. Muskhelishvili [93], Sec. 57)

$$\left.\begin{aligned}\Phi_1(z)&=-\frac{p+iq}{2\pi(1+\varkappa)}\,\frac{1}{z-z_0}+\Phi(z),\\ \Psi_1(z)&=\frac{\varkappa(p-iq)}{2\pi(1+\varkappa)}\,\frac{1}{z-z_0}-\frac{\bar{z}_0(p+iq)}{2\pi(1+\varkappa)}\,\frac{1}{(z-z_0)^2}+\Psi(z),\end{aligned}\right\} \tag{38.25}$$

where Φ, Ψ are the unknown functions which are everywhere holomorphic, with the exception of the points of the real axis, and vanish at infinity.

We substitute the right-hand sides of expressions (38.25) for Φ and Ψ in the boundary conditions (38.10) with $f_0(t)=0$. Then, for the determination of the new unknown functions Φ and Ψ we come, as before, to relation (38.24) and the boundary problem (38.15), where the free term $f(t)$ is defined by the equality

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* Such integrals have been used in the derivation of the basic equation (38.18).

$$f(t)=\frac{iK_0}{2\pi(1+\varkappa)^2}\operatorname{Re}\left\{\frac{\varkappa R}{(t-z_0)^2}+\frac{\varkappa R-2\bar{R}}{(t-\bar{z}_0)^2}+\frac{2\bar{R}(t-z_0)}{(t-\bar{z}_0)^3}\right\}, \quad (38.26)$$

and $R=p+iq$.

In the case under consideration $\mu(x)=0$ and $\nu(x)$ is determined by formula (38.22) with

$$g(t)=-if(t),$$

where $f(t)$ is given by equality (38.26).

2°. Our basic problem posed at the beginning of the section is to determine the elastic equilibrium of a plate with a stiffener and a crack from the boundary conditions

$$\left.\begin{array}{l} h\left[\tau_{xy}^{+}-\tau_{xy}^{-}+i\left(\sigma_y^{+}-\sigma_y^{-}\right)\right]+E_0S_0\dfrac{d^2u^+}{dt^2}=0,\\ u^{+\prime}+iv^{+\prime}=u^{-\prime}+iv^{-\prime} \quad \text{on} \quad L, \end{array}\right\} \quad (38.27)$$

$$\left.\begin{array}{l} \sigma_x^{+}+i\tau_{xy}^{+}=h_1(t),\\ \sigma_x^{-}+i\tau_{xy}^{-}=h_2(t) \quad \text{on} \quad l, \end{array}\right\} \quad (38.28)$$

where h_1, h_2 are given forces applied to the faces of the cut. Without loss of generality, it may be assumed that

$$h_1(t)=h_2(t)=h(t).$$

In terms of complex function theory this problem consists in finding functions $\Phi(z)$, $\Psi(z)$ which are holomorphic in any finite region of the z plane not containing the points of L and l, vanish at infinity and satisfy the boundary conditions

$$\left.\begin{array}{l} S_y^{+}(t,\bar{t})-S_y^{-}(t,\bar{t})-iK_0\operatorname{Re}\left\{\dfrac{d}{dt}D^{+}(t,\bar{t})\right\}=0,\\ D^{+}(t,\bar{t})=D^{-}(t,\bar{t}) \quad \text{on} \quad L, \end{array}\right\} \quad (38.29)$$

$$S_x^{\pm}(t,\bar{t})=h(t) \text{ on } l, \quad (38.30)$$

where use has been made of the notation

$$\left.\begin{array}{l} S_x(t,\bar{t})=\Phi(t)+\overline{\Phi(t)}-t\overline{\Phi'(t)}-\overline{\Psi(t)},\\ S_y(t,\bar{t})=\Phi(t)+\overline{\Phi(t)}+t\overline{\Phi'(t)}+\overline{\Psi(t)},\\ D(t,\bar{t})=\varkappa\Phi(t)-\overline{\Phi(t)}-t\overline{\Phi'(t)}-\overline{\Psi(t)}. \end{array}\right\} \quad (38.31)$$

3°. The solution of a problem with no stiffener, i.e., of a problem involving an isolated crack in an infinite plate, can easily be constructed by N. I. Muskhelishvili's method which leads to a problem of linear relationship ([93], Sec. 120). This is a problem of finding a pair of functions, Φ_0, Ψ_0, which are holomorphic outside the interval l, vanish at infinity and satisfy conditions (38.30).

We introduce into consideration a new sectionally holomorphic function Ω_0 by the equality

$$\Omega_0(z) = \overline{\Phi}_0(-z) - z\overline{\Phi}_0'(-z) - \overline{\Psi}_0(-z). \tag{38.32}$$

As is readily seen, the following formula is then valid:

$$\sigma_x + i\tau_{xy} = \Phi_0(z) + \Omega_0(-\bar{z}) - (z + \bar{z})\,\overline{\Phi'(z)},$$

and in consequence conditions (38.30) become

$$\left.\begin{aligned} \Phi_0^+(t) + \Omega_0^-(t) &= h(t), \\ \Phi_0^-(t) + \Omega_0^+(t) &= h(t) \text{ on } l. \end{aligned}\right\} \tag{38.33}$$

In the case under consideration the resultant of the forces applied to l is zero and, in addition, according to the condition of the problem the functions Φ_0, Ψ_0 (and hence Ω_0) must vanish at infinity.

With these requirements, we find for the solution of problem (38.33)

$$\Phi_0(z) = \frac{1}{2\pi i X(z)} \int\limits_l \frac{X(t)\,h(t)\,dt}{t-z}, \tag{38.34}$$

$$\Omega_0(z) \equiv \Phi_0(z), \tag{38.35}$$

where

$$X(z) = \sqrt{(z-ai)(z-bi)}.$$

By $X(z)$ we understand that branch of the radical which has, for large $|z|$, the form

$$X(z) = z + \alpha_0 + \frac{\alpha_{-1}}{z} + \ldots$$

and by $X(t)$ the value assumed by this function on the left side of the cut.

After Φ_0 is found, the function Ψ_0 is determined according to (38.32) by the formula

$$\Psi_0(z) = \Phi_0(z) - \overline{\Phi}_0(-z) + z\Phi_0'(z). \tag{38.36}$$

In the particular but practically important case when a cut with force-free edges is in a uniform field characterized by constants Γ and Γ' ($\Gamma = \Gamma'$) one should take

$$h(t) = -2\Gamma + \overline{\Gamma}'$$

and the corresponding solution is of the form

$$\left.\begin{aligned} \Phi_0(z) &= \frac{2\Gamma - \overline{\Gamma}'}{2X(z)}\left[z - \frac{(a+b)\,i}{2}\right] + \frac{\overline{\Gamma}'}{2}, \\ \Omega_0(z) &= \frac{2\Gamma - \overline{\Gamma}'}{2X(z)}\left[z - \frac{(a+b)\,i}{2}\right] - \frac{\overline{\Gamma}'}{2}. \end{aligned}\right\} \tag{38.37}$$

4°. The availability of explicit solutions for the foregoing boundary value problems of the special kind makes it possible now to reduce our problem to integral equations with continuous kernels and to construct an analogue of the generalized Schwarz algorithm for its solution.

The left-hand sides of (38.30) are denoted by $u(t)$ and the corresponding solution of this problem by Φ_0, Ψ_0. According to formulas (38.34) and (38.35) we have

$$\left.\begin{aligned} \Phi_0(z) &= \frac{1}{2\pi i X(z)} \int\limits_l \frac{X(t)\,u(t)\,dt}{t-z}, \\ \Psi_0(z) &= \Phi_0(z) - \overline{\Phi}_0(-z) + z\Phi_0'(z). \end{aligned}\right\} \tag{38.38}$$

Similarly, denoting the left-hand side of the first equality of (38.29) by $iv(t)$, where v is a real function on the axis,* and the corresponding solution of problem (38.29) by Φ_1, Ψ_1, by virtue of formulas (38.17), (38.22), (38.24) we find

$$\left.\begin{aligned} \Phi_1(z) &= \frac{1}{2\pi} \int\limits_L \frac{\Omega[v;\,t]}{t-z}\,dt, \\ (\varkappa+1)\,\Omega[v;\,t] &= \int\limits_L v(t-x)\,\eta(x)\,dx + \\ &\quad + \frac{\lambda}{\pi} \int\limits_L v(t-y)\,dy \int\limits_L \frac{\eta'(x)\,dx}{x-y}, \\ \Psi_1(z) &= -\varkappa\overline{\Phi}_1(z) - \Phi_1(z) - z\Phi_1'(z). \end{aligned}\right\} \tag{38.39}$$

The solution of our problem is now represented as

$$\Phi(z) = \Phi_0(z) + \Phi_1(z), \qquad \Psi(z) = \Psi_0(z) + \Psi_1(z), \tag{38.40}$$

where the functions involved in the sums are given by formulas (38.38) and (38.39) with the unknowns u and v.

Substitution of functions (38.40) in the boundary conditions of the problem (38.29), (38.30), followed by elementary computations, gives integral equations for u and v of the form

$$\left.\begin{aligned} & u(t) + \int\limits_L m(t,\,\tau)\,\Omega[v(\tau);\,\tau]\,d\tau = f(t) \text{ on } l, \\ & v(t) - \operatorname{Re} \int\limits_l \left\{ \frac{\partial}{\partial t}[k(t,\,\tau)]\,u(\tau) + \right. \\ & \qquad\qquad \left. + \frac{\partial}{\partial t}[k^*(t,\,\tau)]\,\overline{u(\tau)} \right\} d\tau = 0 \text{ on } L, \end{aligned}\right\} \tag{38.41}$$

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* It is easily verified that in our case this left-hand side is in fact a purely imaginary quantity on the axis.

where

$$\left.\begin{aligned} m(t,\ \tau) &= \frac{1}{2\pi}\left[\frac{(\varkappa+1)\,\tau-(\varkappa-1)\,t}{\tau^2-t^2}+\frac{2\tau}{(\tau+t)^2}\right],\\ k(t,\ \tau) &= \frac{K_0}{2\pi i}\,\frac{X(\tau)}{X(t)}\,\frac{(\varkappa+1)\,\tau+(\varkappa-1)\,t}{\tau^2-t^2},\\ k^*(t,\ \tau) &= \frac{K_0}{2\pi i}\,\frac{\overline{X(\tau)}}{\overline{X(t)}}\left[\frac{(a+b)\,it-2ab}{(t+ai)\,(t+bi)}-\frac{2t}{\tau+t}\right]\frac{1}{\tau+t}. \end{aligned}\right\}\qquad(38.42)$$

When the solution u, v of system (38.41) has been found, the sums of the potentials (38.40) defined by the preceding formulas in terms of u and v give the solution of the boundary value problem at hand.

The stresses in the plate and the elastic displacements of its points are determined according to the well-known Kolosov-Muskhelishvili formulas, and the (normal) stress at the section of the stiffener is

$$\sigma_x^0=\frac{E_0\,(1+\nu)}{E}\,\mathrm{Re}\,\{\varkappa\Phi^+(t)-\overline{\Phi^+(t)}-t\overline{\Phi'^+(t)}-\overline{\Psi^+(t)}\}.\qquad(38.43)$$

It is possible to apply to system (38.41) the method of successive approximations which leads to recurrence relations

$$\left.\begin{aligned} &u_0(t)=h(t),\qquad v_0(t)=0,\\ &u_n(t)=\int\limits_L m(t,\ \tau)\,\Omega\,[v_{n-1};\ \tau]\,d\tau,\\ &v_n(t)=-\mathrm{Re}\int\limits_l\left\{\frac{\partial}{\partial t}\,[k(t,\ \tau)]\,u_{n-1}(\tau)+\right.\\ &\qquad\qquad\left.+\frac{\partial}{\partial t}\,[k^*(t,\ \tau)]\,\overline{u_{n-1}(\tau)}\right\}d\tau\\ &(n=1,\ 2,\ \ldots). \end{aligned}\right\}\qquad(38.44)$$

These relations in turn lead to a certain analogue of Schwarz's algorithm and the successive approximations to Φ_k, Ψ_k are defined by the formulas

$$\left.\begin{aligned} \Phi_0(z) &= \frac{1}{2\pi i X(z)}\int\limits_l\frac{X(t)\,h(t)\,dt}{t-z},\\ \Phi_{2k}(z) &= \frac{1}{2\pi i X(z)}\int\limits_l\frac{X(t)\,h_{2k-1}(t)}{t-z}\,dt\qquad(k=1,\ 2,\ \ldots), \end{aligned}\right\}\qquad(38.45')$$

$$\left.\begin{aligned} \Psi_{2k}(z) &= \Phi_{2k}(z)-\overline{\Phi}_{2k}(-z)+z\Phi_k'(z)\qquad(k=0,\ 1,\ \ldots),\\ \Phi_{2k+1}(z) &= \frac{1}{2\pi}\int\limits_L\frac{\Omega\,[h_{2k};\ t]}{t-z}\,dt,\\ \Psi_{2k+1}(z) &= -\varkappa\overline{\Phi}_{2k+1}(z)-z\Phi_{2k+1}'(z)-\Phi_{2k+1}(z)\\ &\qquad\qquad(k=0,\ 1,\ \ldots); \end{aligned}\right\}\qquad(38.45'')$$

$$\left.\begin{aligned} h_{2k-1}(t) &= \Phi_{2k-1}(t) + \overline{\Phi_{2k-1}(t)} - t\overline{\Phi'_{k-1}(t)} - \overline{\Psi_{2k-1}(t)}, \\ h_{2k}(t) &= -K_0 \operatorname{Re}\left\{\frac{d}{dt}\left[\varkappa\Phi_{2k}(t) - \overline{\Phi_{2k}(t)} - \right.\right. \\ &\qquad\qquad \left.\left. - t\overline{\Phi'_{2k}(t)} - \overline{\Psi_{2k}(t)}\right]\right\}. \end{aligned}\right\} \quad (38.45''')$$

The solution of the problem Φ, Ψ by this method is represented in the form of series

$$\left.\begin{aligned} \Phi(z) &= \Phi_0(z) - \Phi_1(z) + \Phi_2(z) - \ldots, \\ \Psi(z) &= \Psi_0(z) - \Psi_1(z) + \Psi_2(z) - \ldots\,. \end{aligned}\right\} \quad (38.46)$$

Consider the following functional series on l and L composed of the successive approximations (38.44):

$$\left.\begin{aligned} & u_0(t) + u_1(t) + \ldots + u_n(t) + \ldots, \\ & v_1(t) + v_2(t) + \ldots + v_n(t) + \ldots\,. \end{aligned}\right\} \quad (38.47)$$

The functions u_k and v_k are supposed to belong, respectively, to the Hilbert spaces $L_2(a, b)$ and $L_2(-\infty, \infty)$, where the norms of elements are defined by the equalities

$$\|u\| = \left\{\int_a^b u(it)\overline{u(it)}\,dt\right\}^{1/2}, \quad \|v\| = \left\{\int_{-\infty}^{\infty} |v(t)|^2\,dt\right\}^{1/2}. \quad (38.48)$$

Elementary estimates of the operators on the right-hand sides of relations (38.44) allow us to state that numerical series composed of the norms of elements of series (38.47) will be majorized by geometric progressions whose denominator is obviously less than unity for a sufficiently large a. Consequently, for such values of the number a series (38.47) are convergent on the corresponding contours on the average and their sums u and v represent functions continuous on l and L. Series (38.46) which give the solution of the problem uniformly converge in any closed region not containing points of l or L.

III. ON STRESS DISTRIBUTION ROUND HOLES

The starting point of the reasoning of this part is the proposition that of particular interest in the problem of stresses in a plane with a hole is the determination of stresses on the contour of the hole, while the picture itself of the equilibrium state within the region is quite immaterial from the practical point of view.*

It is worth while, therefore, to pose the problem of determining these contour stresses from given elements of the required equilibrium

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* It is assumed, of course, that the medium is homogeneous and the stress components are continuous up to the boundary.

(external forces, elastic displacements of points of the contour, etc.) without actually solving the corresponding problems of the theory of elasticity, i.e., without finding the complex potentials $\varphi(z)$ and $\psi(z)$ in the region occupied by the elastic medium.

On the other hand, on the basis of the well-known formulas for the general complex representation of elastic fields the boundary values of the potentials can be determined from the given values of the stress components on the boundary and then the potentials φ and ψ can be determined inside the region from their boundary values, and conversely; the functions $\varphi(z)$ and $\psi(z)$, after their boundary values are found, completely determine the boundary values of all three stress components. It is, therefore, of course only a question of constructing an algorithm (for the determination of the contour stresses) most convenient from the practical point of view.

The following shows the possibility of realizing the outlined approach.*

39. First Fundamental Problem

Consider an elastic medium under conditions of plane strain or generalized plane stress which fills all of the infinite plane of the variable z and is weakened by a hole of arbitrary shape. By convention, the first fundamental problem refers to the determination of the stress components along the contour of the hole when the external forces are prescribed on the same contour. The state of stress at infinity is, of course, also given.

We shall proceed from the complex representation formulas (1.6):

$$\left.\begin{aligned} \sigma_\rho^{(1)}+\sigma_\vartheta^{(1)} &= 2\left[\Phi_1(\zeta)+\overline{\Phi_1(\zeta)}\right], \\ \sigma_\vartheta^{(1)}-\sigma_\rho^{(1)}+2i\tau_{r\vartheta}^{(1)} &= \frac{2\zeta^2}{\rho^2\overline{\omega'(\zeta)}}\left[\overline{\omega(\zeta)}\,\Phi_1'(\zeta)+\omega'(\zeta)\,\Psi_1(\zeta)\right], \end{aligned}\right\} \tag{39.1}$$

where $\sigma_\rho^{(1)}$, $\sigma_\vartheta^{(1)}$, $\tau_{r\vartheta}^{(1)}$ are the components of stress in curvilinear co-ordinates related to the conformal mapping

$$z=\omega(\zeta), \qquad \zeta=\rho e^{i\vartheta}, \tag{39.2}$$

$$\left.\begin{aligned} \Phi_1(\zeta) &= \frac{\varphi_1'(\zeta)}{\omega'(\zeta)}, & \Psi_1(\zeta) &= \frac{\psi_1'(\zeta)}{\omega'(\zeta)}, \\ \varphi_1(\zeta) &= \varphi[\omega(\zeta)], & \psi_1(\zeta) &= \psi[\omega(\zeta)], \end{aligned}\right\} \tag{39.3}$$

and $\varphi(z)$, $\psi(z)$, as always, denote the complex potentials in the z plane.

Suppose that the region Σ with boundary γ which is mapped by relation (39.2) onto the physical region S represents an infinite region in the plane of the auxiliary variable ζ outside the unit circle centred at the origin. This means that the function $\omega(\zeta)$ is representable in the region Σ as the series

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* The subject of stress distribution round holes is treated in the well-known monograph by G. N. Savin [107] (see also the second edition [110]).

$$\omega(\zeta)=R\zeta+\frac{c_1}{\zeta}+\frac{c_2}{\zeta^2}+\ldots+\frac{c_n}{\zeta^n}+\ldots . \tag{39.4}$$

It may also be recalled that the functions $\varphi_1(\zeta)$, $\psi_1(\zeta)$ admit in the same region the representation (N. I. Muskhelishvili [93] Sec. 50)

$$\left.\begin{aligned}\varphi_1(\zeta)&=-\frac{X+iY}{2\pi(1+\varkappa)}\ln\zeta+R\Gamma\zeta+\varphi_*(\zeta),\\ \psi_1(\zeta)&=\frac{\varkappa(X-iY)}{2\pi(1+\varkappa)}\ln\zeta+R\Gamma'\zeta+\psi_*(\zeta),\end{aligned}\right\} \tag{39.5}$$

where φ_*, ψ_* are holomorphic functions, regular at $\zeta=\infty$, and X, Y are the components of the resultant of the external forces applied to the boundary of the region S.

Assume now

$$\sigma_\rho^{(1)}=\sigma_\rho+\sigma_\rho^0,\qquad \sigma_\vartheta^{(1)}=\sigma_\vartheta+\sigma_\vartheta^0,\qquad \tau_{r\vartheta}^{(1)}=\tau_{r\vartheta}+\tau_{r\vartheta}^0, \tag{39.6}$$

where σ_ρ^0, σ_ϑ^0, $\tau_{r\vartheta}^0$ are the components of stress in a uniform field characterized by the constants Γ and Γ' ($\Gamma=\overline{\Gamma}$), and represent relations (39.1) with $\rho=1$ as follows ($\sigma=e^{i\vartheta}$):

$$\begin{gathered}2\left[\Phi(\sigma)+\overline{\Phi(\sigma)}\right]=\sigma_\rho+\sigma_\vartheta \text{ on } \gamma,\\ 2\left[\overline{\omega(\sigma)}\,\Phi'(\sigma)+\omega'(\sigma)\,\Psi(\sigma)\right]=\sigma^2\overline{\omega'(\sigma)}\left[\sigma_\vartheta-\sigma_\rho+2i\tau_{\rho\vartheta}\right].\end{gathered} \tag{39.7}$$

Here Φ and Ψ are unknown functions, holomorphic in the region Σ, and, by formulas (39.3) to (39.5), having for large $|\zeta|$ the form

$$\Phi(\zeta)=\frac{A}{\zeta}+O(\zeta^{-2}),\qquad \Psi(\zeta)=\frac{A'}{\zeta}+O(\zeta^{-2}), \tag{39.8}$$

$$A=-\frac{X+iY}{2\pi R(1+\varkappa)},\qquad A'=\frac{\varkappa(X-iY)}{2\pi R(1+\varkappa)}. \tag{39.9}$$

The stresses σ_ρ^0, σ_ϑ^0, $\tau_{\rho\vartheta}^0$ are calculated from the formulas

$$\left.\begin{aligned}\sigma_\rho^0+\sigma_\vartheta^0&=2\left[\Phi_0(\zeta)+\overline{\Phi_0(\zeta)}\right],\\ \sigma_\vartheta^0-\sigma_\rho^0+2i\tau_{\rho\vartheta}^0&=\frac{2\zeta^2}{\rho^2\,\overline{\omega'(\zeta)}}\left[\overline{\omega(\zeta)}\,\Phi_0'(\zeta)+\omega'(\zeta)\,\Psi_0(\zeta)\right]\end{aligned}\right\} \tag{39.10}$$

with

$$\Phi_0(\zeta)=\frac{R\Gamma}{\omega'(\zeta)},\qquad \Psi_0(\zeta)=\frac{R\Gamma'}{\omega'(\zeta)}. \tag{39.11}$$

In accordance with formulas (39.6) we have

$$\Phi_1(\zeta)=\Phi(\zeta)+\Phi_0(\zeta),\qquad \Psi_1(\zeta)=\Psi(\zeta)+\Psi_0(\zeta).$$

It is well known that, whenever the region occupied by the elastic medium contains the infinitely remote point of the plane, the constants Γ, Γ', X, Y are specified by the problem itself. The stresses σ_ρ^0, σ_ϑ^0, $\tau_{r\vartheta}^0$ and the constants A and A' are therefore known, they are defined by formulas (39.9) and (39.10).

Our problem is to determine the normal stress σ_ϑ by formulas (39.7) from the given values on γ of the other two components, σ_ρ and $\tau_{\rho\vartheta}$. Assume

$$\sigma_\vartheta + \sigma_\rho = \Omega(\sigma), \qquad -2(\sigma_\rho - i\tau_{\rho\vartheta}) = \Theta(\sigma) \text{ on } \gamma \tag{39.12}$$

and rewrite (39.7) as

$$\left.\begin{aligned} 2[\Phi(\sigma)+\overline{\Phi(\sigma)}] &= \Omega(\sigma), \\ \overline{\sigma^2\omega'(\sigma)}\,[\Omega(\sigma)+\Theta(\sigma)] - 2\overline{\omega(\sigma)}\,\Phi'(\sigma) &= 2\omega'(\sigma)\,\Psi(\sigma) \text{ on } \gamma. \end{aligned}\right\} \tag{39.13}$$

On the basis of the first of the preceding equalities we represent the function $\Phi(\zeta)$ with the aid of the Schwarz integral. By formula (39.8) we have

$$2\Phi(\zeta) = -\frac{1}{2\pi i}\int_\gamma \frac{\Omega(\sigma)\,d\sigma}{\sigma-\zeta} \qquad (\zeta \text{ in } \Sigma^-), \tag{39.14}$$

where, for convenience, we denote by Σ^- the region Σ, and by Σ^+ the region of the unit circle $|\zeta| < 1$. From this we have

$$2\Phi'(\zeta) = -\frac{1}{2\pi i}\int_\gamma \frac{\Omega'(\sigma)\,d\sigma}{\sigma-\zeta} \qquad \left(\Omega'(\sigma) = \frac{d\Omega(\sigma)}{d\sigma}\right), \tag{39.15}$$

$$2\Phi'(\sigma) = \frac{\Omega'(\sigma)}{2} - \frac{1}{2\pi i}\int_\gamma \frac{\Omega'(\tau)\,d\tau}{\tau-\sigma} \text{ on } \gamma, \tag{39.16}$$

where τ, just as σ, denotes the affix of a point lying on the circumference γ.

Substituting expressions (39.16) in the second equality of (39.13), we obtain

$$\overline{\sigma^2\omega'(\sigma)}\,[\Omega(\sigma)+\Theta(\sigma)] - \frac{\overline{\omega(\sigma)}\,\Omega'(\sigma)}{2} + \\ + \frac{\overline{\omega(\sigma)}}{2\pi i}\int_\gamma \frac{\Omega'(\tau)\,d\tau}{\tau-\sigma} = 2\omega'(\sigma)\,\Psi(\sigma). \tag{39.17}$$

The right-hand side of this equality [for the present we denote it by $F(\sigma)$] represents the boundary values on γ of a function of ζ, holomorphic in Σ^- and vanishing, by virtue of formulas (39.4) and (39.8), at infinity. On the basis of the well-known proposition of complex function theory we have therefore (see, for example, N. I. Muskhelishvili [93], Sec. 73)

$$\frac{1}{2\pi i}\int_\gamma \frac{F(\sigma)\,d\sigma}{\sigma-\zeta} = 0 \text{ for all } \zeta \text{ from } \Sigma^+,$$

or, what is equivalent,

$$F(\sigma)+\frac{1}{\pi i}\int\limits_{\gamma}\frac{F(\tau)\,d\tau}{\tau-\sigma}=0 \text{ on } \gamma. \qquad (39.18)$$

Equalities (39.17) and (39.18) together give

$$\overline{\sigma^2\omega'(\sigma)}\,[\Omega(\sigma)+\Theta(\sigma)]-\frac{\overline{\omega(\sigma)}\,\Omega'(\sigma)}{2}+\frac{\overline{\omega(\sigma)}}{2\pi i}\int\limits_{\gamma}\frac{\Omega'(\tau)\,d\tau}{\tau-\sigma}+$$
$$+\frac{1}{\pi i}\int\limits_{\gamma}\left\{\overline{\tau^2\omega'(\tau)}\,[\Omega(\tau)+\Theta(\tau)]-\frac{\overline{\omega(\tau)}\,\Omega'(\tau)}{2}+\right.$$
$$\left.+\frac{\overline{\omega(\tau)}}{2\pi i}\int\limits_{\gamma}\frac{\Omega'(\sigma_1)\,d\sigma_1}{\sigma_1-\tau}\right\}\frac{d\tau}{\tau-\sigma}=0. \qquad (39.19)$$

To the composition of integrals in (39.19) understood as the principal value of the Cauchy integrals we apply the Poincaré-Bertrand interchange formula (see, for example, N. I. Muskhelishvili [92], Sec. 28). We have

$$\frac{1}{\pi i}\int\limits_{\gamma}\frac{\overline{\omega(\tau)}\,d\tau}{\tau-\sigma}\left\{\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\Omega'(\sigma_1)\,d\sigma_1}{\sigma_1-\tau}\right\}=\frac{\overline{\omega(\sigma)}\,\Omega'(\sigma)}{2}+$$
$$+\frac{1}{\pi i}\int\limits_{\gamma}\Omega'(\sigma_1)\,d\sigma_1\left\{\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\overline{\omega(\tau)}\,d\tau}{(\tau-\sigma)(\sigma_1-\tau)}\right\}. \qquad (39.20)$$

The second term on the right-hand side of this equality can be considerably simplified. Denoting this term, for the time being, by J and taking into account that

$$\frac{1}{(\tau-\sigma)(\sigma_1-\tau)}=\frac{1}{\sigma_1-\sigma}\left[\frac{1}{\tau-\sigma}-\frac{1}{\tau-\sigma_1}\right],$$

we can write

$$J=\frac{1}{\pi i}\int\limits_{\gamma}\frac{\Omega'(\sigma_1)\,d\sigma_1}{\sigma_1-\sigma}\left\{\frac{1}{2\pi i}\int\limits_{\gamma}\overline{\omega(\tau)}\left[\frac{1}{\tau-\sigma}-\frac{1}{\tau-\sigma_1}\right]d\tau\right\}. \qquad (39.21)$$

We observe now that, by formula (39.4), the function $\omega(\zeta)$, holomorphic in the region Σ^-, has a principal part equal to $R\zeta$ at infinity. The following formula is therefore valid (N. I. Muskhelishvili [93], p. 252):

$$\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\overline{\omega(\tau)}\,d\tau}{\tau-\zeta}=\bar{\omega}\left(\frac{1}{\zeta}\right)-\frac{R}{\zeta} \qquad (\zeta \text{ in } \Sigma^+).$$

A limiting process when $\zeta\to\sigma$ gives

$$\frac{\overline{\omega(\sigma)}}{2}+\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\overline{\omega(\tau)}\,d\tau}{\tau-\sigma}=\overline{\omega(\sigma)}-\frac{R}{\sigma},$$

from which

$$\frac{1}{2\pi i}\int_{\gamma}\frac{\overline{\omega(\tau)}\,d\tau}{\tau-\sigma}=\frac{\overline{\omega(\sigma)}}{2}-\frac{R}{\sigma}\quad\text{on } \gamma. \tag{39.22}$$

The preceding formula makes it possible to give expression (38.21) the form

$$J=\frac{1}{2\pi i}\int_{\gamma}\Omega'(\tau)\left[\frac{\overline{\omega(\sigma)}-\overline{\omega(\tau)}}{\tau-\sigma}-\frac{2R}{\sigma\tau}\right]d\tau.$$

But according to our notation

$$\Omega(\sigma)=2\left[\Phi(\sigma)+\overline{\Phi(\sigma)}\right]\ \text{on } \gamma,$$

where $\Phi(\zeta)$ is a holomorphic function in Σ^- which has the form (39.8) for large $|\zeta|$. From this, by using the residue theorem, we easily find

$$\frac{1}{2\pi i}\int_{\gamma}\frac{\Omega'(\tau)\,d\tau}{\tau}=\frac{1}{\pi i}\int_{\gamma}\frac{d\left[\Phi(\tau)+\overline{\Phi(\tau)}\right]}{\tau}=\frac{1}{\pi i}\int_{\gamma}\frac{\Phi(\tau)+\overline{\Phi(\tau)}}{\tau^2}\,d\tau=2\bar{A}.$$

Hence,

$$J=\frac{1}{2\pi i}\int_{\gamma}\Omega'(\tau)\frac{\overline{\omega(\sigma)}-\overline{\omega(\tau)}}{\tau-\sigma}\,d\tau-\frac{4\bar{A}R}{\sigma}. \tag{39.23}$$

Considering expressions (39.20) and (39.23), we obtain from equality (39.19) the required functional equation for determining $\Omega(\sigma)$ as follows:

$$\overline{\sigma^2\omega'(\sigma)}\left[\Omega(\sigma)+\Theta(\sigma)\right]+\frac{1}{\pi i}\int_{\gamma}\overline{\tau^2\omega'(\tau)}\left[\Omega(\tau)+\Theta(\tau)\right]\frac{d\tau}{\tau-\sigma}-$$
$$-\frac{1}{\pi i}\int_{\gamma}\frac{\overline{\omega(\tau)}-\overline{\omega(\sigma)}}{\tau-\sigma}\,\Omega'(\tau)\,d\tau-\frac{4\bar{A}R}{\sigma}=0\ \text{on } \gamma. \tag{39.24}$$

All known quantities in Eq. (39.24) are transposed to the right-hand side of the equality and, to facilitate the subsequent computations, the equation is rewritten as

$$\lim_{\zeta\to\sigma}\left\{\frac{1}{2\pi i}\int_{\gamma}\frac{\overline{\tau^2\omega'(\tau)}\,\Omega(\tau)}{\tau-\zeta}\,d\tau\right\}-\frac{1}{2\pi i}\int_{\gamma}\frac{\overline{\omega(\tau)}-\overline{\omega(\sigma)}}{\tau-\sigma}\,\Omega'(\tau)\,d\tau=$$
$$=\lim_{\zeta\to\sigma}F(\zeta)+\frac{2\bar{A}R}{\sigma}, \tag{39.25}$$

$$F(\zeta)=-\frac{1}{2\pi i}\int_{\gamma}\frac{\overline{\tau^2\omega'(\tau)}\,\Theta(\tau)}{\tau-\zeta}\,d\tau. \tag{39.26}$$

It is supposed that the point ζ approaches a point σ on γ remaining all the time in the region Σ^+.

Assume on γ

$$\Omega(\sigma) = \sum_{-\infty}^{\infty} a_k \sigma^k \quad (a_{-k} = \bar{a}_k, \qquad k = 0, 1, \ldots), \qquad (39.27)$$

$$-\overline{\sigma^2 \omega'(\sigma)}\, \Theta(\sigma) = \sum_{-\infty}^{\infty} A_k \sigma^k. \qquad (39.28)$$

The coefficients A_k are known numbers and a_k are not known.

From expression (39.4) we have

$$\overline{\sigma^2 \omega'(\sigma)} = \frac{R}{\sigma^2} - \sum_{k=0}^{\infty} (k+1)\, \bar{c}_{k+1} \sigma^k. \qquad (39.29)$$

Multiplication of the Fourier series (39.27) and (39.29) gives

$$\overline{\sigma^2 \omega'(\sigma)}\, \Omega(\sigma) = \sum_{m=0}^{\infty} \left[R a_{m+2} - \sum_{k=1}^{\infty} k \bar{c}_k a_{m-k+1}\right] \sigma^m + \ldots, \qquad (39.30)$$

where the terms containing negative powers of the argument σ are not written out.

Of course, we assume that the operation of multiplying the series performed above is legitimate, as also is the termwise integration of series (39.30) under the Cauchy-type integral. Then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{\overline{\sigma^2 \omega'(\sigma)}\, \Omega(\sigma)\, d\sigma}{\sigma - \zeta} = \sum_{m=0}^{\infty} \left[R a_{m+2} - \sum_{k=1}^{\infty} k \bar{c}_k a_{m-k+1}\right] \zeta^m \qquad (\zeta \text{ in } \Sigma^+). \qquad (39.31)$$

From formula (39.4) we have further

$$\frac{\overline{\omega(\tau)} - \overline{\omega(\sigma)}}{\tau - \sigma} = -\frac{R}{\tau\sigma} + \sum_{m=0}^{\infty} \left\{ \sum_{k=0}^{\infty} \bar{c}_{m+k+1} \tau^k \right\} \sigma^m. \qquad (39.32)$$

By multiplying this series by the Fourier series

$$\Omega'(\tau) = \sum_{-\infty}^{\infty} k a_k \tau^{k-1}$$

and integrating the product with respect to τ along γ, we obtain, provided the termise integration of the series is legitimate, the formula

$$-\frac{1}{2\pi i} \int_{\gamma} \frac{\overline{\omega(\tau)} - \overline{\omega(\sigma)}}{\tau - \sigma}\, \Omega'(\tau)\, d\tau =$$

$$= \frac{R a_1}{\sigma} + \sum_{m=0}^{\infty} \left\{ \sum_{k=1}^{\infty} k \bar{c}_{m+k+1} a_{-k} \right\} \sigma^m \text{ on } \gamma \qquad (39\ 33)$$

Inserting expressions (39.31) and (39.33) in Eq. (39.25) and comparing the coefficients of all powers of σ on the two sides of the equation, we obtain the following set of equalities:*

$$a_1 = 2\bar{A}, \qquad R a_{m+2} - \sum_{k=1}^{m} (m-k+1)\,\bar{c}_{m-k+1} a_k -$$

$$-(m+1)\sum_{k=1}^{\infty} \bar{c}_{m+k+1}\bar{a}_k = A_m \quad (a_{-k} = \bar{a}_k, \quad m = 0,\ 1,\ 2,\ \ldots). \quad (39.34)$$

These equalities give an infinite system of linear algebraic equations for determining the unknown coefficients a_k ($k = 1, 2, \ldots$). Its solution a_k [that guarantees the legitimacy of the assumptions made above in the derivation of Eqs. (39.34)] determines $\Omega(\sigma)$ in the form of (39.27), which leads to the solution of our problem. The unknown tensile stress $\sigma_{\vartheta}^{(1)}$ is given by the formula

$$\sigma_{\vartheta}^{(1)} = \Omega(\sigma) + \Omega_0(\sigma) - \sigma_{\rho}^{(1)} \qquad (\Omega_0(\sigma) = \sigma_{\vartheta}^{0} + \sigma_{\rho}^{0}). \quad (39.35)$$

The application of the foregoing method will be illustrated below by the familiar examples worked out in detail (by solving the corresponding plane problems of elasticity) in the many times mentioned monograph by N. I. Muskhelishvili [93].

Note 1. By using conformal mapping and power series, N. I. Muskhelishvili reduced the plane problem for an arbitrary singly connected region to an infinite system of linear algebraic equations in the unknown coefficients of the series expansion of the function $\varphi_1(\zeta)$ inside (or outside) the unit circle (N. I. Muskhelishvili [93], Sec. 63). Subsequently, this procedure of solving the plane problem has found wide application and formed the basis of one of the methods known at present as Muskhelishvili's.**

The infinite system (39.34) constructed above for the Fourier coefficients of the boundary values of the real part of the potential $\Phi_1(\zeta)$ outwardly resembles N. I. Muskhelishvili's system but is essentially different from it.

The main feature of N. I. Muskhelishvili's system is that in the case of mapping with a finite number of terms, i.e., when

$$\omega(\zeta) = R\zeta + \frac{c_1}{\zeta} + \frac{c_2}{\zeta^2} + \ldots + \frac{c_n}{\zeta^n},$$

---

* The unknown a_0 is zero. This is easily verified by integrating the equality

$$2\left[\frac{\Phi(\sigma)}{\sigma} + \frac{\overline{\Phi(\sigma)}}{\sigma}\right] = \frac{\Omega(\sigma)}{\sigma}$$

along the circumference γ and noting the first of equalities (39.8).

** In Chap. II of the present book it was referred to as Method 2 (of N. I. Muskhelishvili).

where n is some natural number, it breaks off and reduces to a finite system of linear equations which is always solvable (under certain conditions of statics). System (39.34), however, has no such advantage. To be more accurate, in the indicated case this system breaks off in only one direction, i.e., each of its equations contains a finite number of unknowns (the same number for all equations), but the number of equations in the system always remains infinite.

To make mention of a certain advantage of system (39.34), however, it should be noted that the system is formed in a very elementary way; the finding of the coefficients of the unknowns in the system requires nothing but assigning a mapping function in the form of (39.4).

It would be of some interest, in our opinion, to investigate system (39.34) in greater detail and to compare more fully the above two systems from the viewpoint of their practical applications.

Note 2. The constants A and A' giving, by formulas (39.8), the first coefficients in the expansions of the functions $\Phi(\zeta)$ and $\Psi(\zeta)$ in the neighbourhood of the point $\zeta = \infty$ need not be assigned in advance. They can be determined from the conditions of the problem itself as follows.

By subtracting equalities (39.1), we obtain on γ

$$\overline{\sigma^2\omega'(\sigma)}\,[\Phi_1(\sigma)+\overline{\Phi_1(\sigma)}]-\overline{\omega(\sigma)}\,\Phi_1'(\sigma)-\omega'(\sigma)\,\Psi_1(\sigma)=$$
$$=\overline{\sigma^2\omega'(\sigma)}\,(\sigma_\rho^{(1)}-i\tau_{\rho\vartheta}^{(1)}). \qquad (39.36)$$

It will be recalled that the functions $\sigma_\rho^{(1)}$ and $\tau_{\rho\vartheta}^{(1)}$ are given on γ and represent the components, referred to curvilinear co-ordinate axes, of the stress vector applied to the boundary of an elastic body, and $\Phi_1(\zeta)$, $\Psi_1(\zeta)$ are holomorphic functions outside γ which have, for large $|\zeta|$, the form

$$\Phi_1(\zeta)=\Gamma+\frac{A}{\zeta}+O(\zeta^{-2}), \quad \Psi_1(\zeta)=\Gamma'+\frac{A_1'}{\zeta}+O(\zeta^{-2}). \qquad (39.37)$$

Here Γ and Γ' are given constants, A and A' are also constants which are to be determined.

We observe now that if X_n, Y_n are the components along the x and y axes of the stress vector applied to the boundary L of the body, the following formula is valid:

$$\sigma_\rho^{(1)}+i\tau_{\rho\vartheta}^{(1)}=\frac{\overline{\sigma\omega'(\sigma)}}{|\omega'(\sigma)|}(X_n+iY_n) \qquad \text{on } \gamma. \qquad (39.38)$$

On multiplying equality (39.36) by $d\sigma$, integrating along γ and taking into account formulas (39.37) and (39.38) we find

$$A-\bar{A}'=-\frac{X+iY}{2\pi R(1+\varkappa)}, \qquad (39.39)$$

where X and Y denote the components of the resultant of the external forces.

A second condition for the determination of A and A' is obtained by requiring that the elastic displacements, referred to curvilinear co-ordinates, be single valued

$$2\mu\left(v_\rho^{(1)}+iv_\vartheta^{(1)}\right)=\frac{\bar{\zeta}}{\rho}\frac{\overline{\omega'(\zeta)}}{|\omega'(\zeta)|}\left[\varkappa\varphi_1(\zeta)-\omega(\zeta)\overline{\Phi_1(\zeta)}-\overline{\psi_1(\zeta)}\right]$$

in the region Σ^-. The condition for the single-valuedness of this expression, as is easily verified, leads to the equality

$$\varkappa A+\bar{A}'=0. \tag{39.40}$$

By solving Eqs. (39.39) and (39.40) simultaneously, we obtain formulas (39.9).

40. Examples

1. Simple (uniaxial) tension of an infinite plate weakened by a circular hole. Let an infinite plate stretched at infinity by forces in the x direction have a circular hole centred at the origin whose edge is free from external forces. The intensity of tensile force is denoted by p and the radius of the hole by R. Then

$$\sigma_\rho^{(1)}=\tau_{\rho\vartheta}^{(1)}=0,\qquad \Gamma=\frac{p}{4},\qquad \Gamma'=-\frac{p}{2},\qquad A=A'=0,$$

$$\omega(\zeta)=R\zeta,\qquad \Phi_0(\zeta)=\frac{p}{4},\qquad \Psi_0(\zeta)=-\frac{p}{2}.$$

Further, on the basis of formulas (39.10), (39.12), (39.26), we have

$$\overline{\sigma^2\omega'(\sigma)}\,\Theta(\sigma)=-2\overline{\sigma^2\omega'(\sigma)}\,(\sigma_\rho-i\tau_{\rho\vartheta})=$$

$$=2\overline{\sigma^2\omega'(\sigma)}\,(\sigma_\rho^0-i\tau_{\rho\vartheta}^0)=pR\left(1+\frac{1}{\sigma^2}\right),$$

$$F(\zeta)=-\frac{Rp}{2\pi i}\int_\gamma\left(1+\frac{1}{\sigma^2}\right)\frac{d\sigma}{\sigma-\zeta}=-Rp\qquad(\zeta\text{ in }\Sigma^+).$$

From this the Fourier coefficients A_n are found to be

$$A_0=-Rp,\qquad A_k=0\qquad(k=1,\ 2,\ \ldots).$$

Solving system (39.34) gives

$$a_2=\bar{a}_{-2}=-p,\qquad \text{all other } a_k=0.$$

Consequently,

$$\Omega(\sigma)=-p\sigma^2-p\bar{\sigma}^2=-2p\cos 2\vartheta,$$

$$\Omega_0(\sigma)=2\left[\Phi_0(\sigma)+\overline{\Phi_0(\sigma)}\right]=p$$

and formula (39.35) yields, finally,

$$\sigma_\vartheta^{(1)}=p\,(1-2\cos 2\vartheta)\ \text{on}\ \gamma. \tag{40.1}$$

2. Uniform normal pressure applied to the contour of a circular hole. Denoting the intensity of pressure by p, we have in the case under consideration

$$\sigma_\rho^{(1)} = -p, \qquad \tau_{\rho\vartheta}^{(1)} = 0 \text{ on } \gamma, \qquad \Gamma = \Gamma' = A = A' = 0,$$

$$\Phi_0(\zeta) = \Psi_0(\zeta) = 0, \quad \Omega_0(\sigma) = 0 \text{ everywhere in } \Sigma^- \text{ and on } \gamma.$$

Further,

$$\overline{\sigma^2 \omega'(\sigma)}\, \Theta(\sigma) = -2R\bar{\sigma}^2 (\sigma_\rho^1 - i\tau_{\rho\vartheta}^{(1)}) = 2pR\bar{\sigma}^2,$$

and hence, by formula (39.26),

$$F(\zeta) = 0 \text{ everywhere in } \Sigma^+.$$

The system of equations (39.34) has a zero solution and formula (39.35) gives

$$\sigma_\vartheta^{(1)} = p \text{ on } \gamma. \tag{40.2}$$

3. Uniform shearing stress applied to the contour of a circular hole. If T is the magnitude of a given shearing stress, the condition of the problem is represented as

$$\sigma_\rho^1 = 0, \ \tau_{\rho\vartheta}^{(1)} = T \text{ on } \gamma, \qquad \Gamma = \Gamma' = A = A' = 0,$$

$$\Omega_0(\sigma) = 0 \qquad \overline{\sigma^2 \omega'(\sigma)}\, \Theta(\sigma) = 2iRT\bar{\sigma}^2.$$

From this, as in the preceding example, we have

$$F(\zeta) = 0 \text{ in } \Sigma^-, \qquad \Omega(\sigma) = 0 \text{ on } \gamma$$

and formula (39.35) gives

$$\sigma_\vartheta^{(1)} = 0 \text{ on } \gamma. \tag{40.3}$$

4. Tension of an infinite plate weakened by an elliptical hole. In this case, too, which is not so elementary as the previous ones, the solution of the problem presents no difficulty.

Suppose that the major axis of the ellipse is located on the x axis and a tensile force of magnitude p is parallel to the y axis at infinity. In the case under consideration

$$\omega(\zeta) = R\left(\zeta + \frac{m}{\zeta}\right) \qquad (R > 0, \ 0 \leqslant m < 1), \tag{40.4}$$

$$\sigma_\rho^{(1)} = \tau_{\rho\vartheta}^{(1)} = 0 \text{ on } \gamma, \qquad \Gamma = \frac{p}{4}, \qquad \Gamma' = \frac{p}{2}. \tag{40.5}$$

Besides, on the basis of formulas (39.11)

$$\Phi_0(\zeta) = \frac{\Gamma\zeta^2}{\zeta^2 - m}, \qquad \Psi_0(\zeta) = \frac{\Gamma'\zeta^2}{\zeta^2 - m}. \tag{40.6}$$

Inserting functions (40.6) into the right side of the equality

$$\overline{\sigma^2 \omega'(\sigma)}\, (\sigma_\rho^0 - i\tau_{\rho\vartheta}^0) =$$

$$= \overline{\sigma^2 \omega'(\sigma)}\, [\Phi_0(\sigma) + \overline{\Phi_0(\sigma)}] - \overline{\omega(\sigma)}\, \Phi_0'(\sigma) - \omega'(\sigma)\, \Psi_0(\sigma$$

which follows from formulas (39.10), we have

$$\overline{\sigma^2\omega'(\sigma)}\,(\sigma_\rho^0 - i\tau_{\rho\vartheta}^0) = R\Gamma\left[\frac{1}{\sigma^2} + \frac{1-m\sigma^2}{\sigma^2-m} + 2m\frac{\sigma^2+m}{(\sigma^2-m)^2}\right] - R\Gamma'. \tag{40.7}$$

Noting that by virtue of the boundary conditions (40.5) $\sigma_\rho^0 - i\tau_{\rho\vartheta}^0 = -(\sigma_\rho - i\tau_{\rho\vartheta})$, from formulas (39.26) and (40.7) the function $F(\zeta)$ is obtained as

$$F(\zeta) = -\frac{1}{\pi i}\int\limits_\gamma \frac{\overline{\sigma^2\omega'(\sigma)}\,(\sigma_\rho^0 - i\tau_{\rho\vartheta}^0)}{\sigma-\zeta}\,d\sigma =$$

$$= 2(m\Gamma R + R\Gamma') = \frac{p}{2}R(m+2). \tag{40.8}$$

From this it follows that

$$A_0 = \frac{PR(m+2)}{2}, \quad A_k = 0 \qquad (k \geqslant 1).$$

The system of equations (39.34) in our case becomes

$$a_{2n-1} = 0 \qquad (n \geqslant 1),$$

$$a_2 = \frac{p(m+2)}{2}, \quad a_4 = ma_2, \quad a_6 = ma_4, \ \ldots, \ a_{2k} = ma_{2k-2}, \ \ldots .$$

Consequently,

$$\Omega(\sigma) = 2\operatorname{Re}\sum_{k=1}^{\infty} a_{2k}\sigma^{2k} = p(m+2)\operatorname{Re}\sum_{k=1}^{\infty} m^{k-1}\sigma^{2k} =$$

$$= p(m+2)\operatorname{Re}\frac{\sigma^2}{1-m\sigma^2} = \frac{p(m+2)}{2}\left[\frac{\sigma^2}{1-m\sigma^2} + \frac{1}{\sigma^2-m}\right]. \tag{40.9}$$

Further we have, from formulas (40.6),

$$\Omega_0(\sigma) = 2\left[\Phi_0(\sigma) + \overline{\Phi_0(\sigma)}\right] = \frac{p}{2}\left[\frac{1}{1-m\sigma^2} + \frac{\sigma^2}{\sigma^2-m}\right]. \tag{40.10}$$

The preceding two equalities determine the required stress $\sigma_\vartheta^{(1)}$ by formula (39.35) as follows:

$$\sigma_\vartheta^{(1)} = p\frac{1-m^2-2(m-\cos 2\vartheta)}{1-2m\cos 2\vartheta + m^2} \qquad (0 \leqslant \vartheta \leqslant 2\pi). \tag{40.11}$$

It is known that the tensile stress $\sigma_\vartheta^{(1)}$ attains its maximum value at the vertices of the ellipse ($\vartheta = 0, \pi$).

From (40.11) we have

$$\max \sigma_\vartheta^{(1)} = \sigma_\vartheta^{(1)}\big|_{\vartheta=0,\,\pi} = p\left[1 + \frac{2(1+m)}{1-m}\right]. \tag{40.12}$$

41. Second Fundamental Problem

We so call the problem of determining the boundary values of the stress components from given values of the elastic displacements along the boundary of a body.

The given conditions are the stresses and rotation at infinity as well as the components X, Y of the resultant of the external forces applied to the contour of the hole.

The consideration in the present section is restricted to the case of plane strain in an elastic body on the assumption of incompressibility of material ($\nu = 1/2$, ν is Poisson's ratio).

Under these conditions the elastic constant $\varkappa$ appearing in the expressions for elastic displacements is equal to 1, and the complete system of complex representation formulas for stresses and displacements in curvilinear co-ordinates is given as ($\sigma = e^{i\vartheta}$)

$$\sigma_\rho^{(1)} + \sigma_\vartheta^{(1)} = 2\left[\Phi_1(\sigma) + \overline{\Phi_1(\sigma)}\right],$$

$$\sigma_\vartheta^{(1)} - \sigma_\rho^{(1)} + 2i\tau_{\rho\vartheta}^{(1)} = \frac{2\sigma^2}{\overline{\omega'(\sigma)}}\left[\overline{\omega(\sigma)}\,\Phi_1'(\sigma) + \omega'(\sigma)\,\Psi_1(\sigma)\right], \qquad (41.1)$$

$$\sigma_\rho^{(1)} - i\tau_{\rho\vartheta}^{(1)} = \Phi_1(\sigma) + \overline{\Phi_1(\sigma)} - \frac{\sigma^2}{\overline{\omega'(\sigma)}}\left[\overline{\omega(\sigma)}\,\Phi_1'(\sigma) + \omega'(\sigma)\,\Psi_1(\sigma)\right], \qquad (41.2)$$

$$2\mu(u_1 + iv_1) = \varphi_1(\sigma) - \omega(\sigma)\,\overline{\Phi_1(\sigma)} - \overline{\psi_1(\sigma)}. \qquad (41.3)$$

By the condition of the problem, the following expression is given on the circumference γ:

$$2\mu(u_1 + iv_1) = W_1(\sigma)$$

from which we have to determine the stresses $\sigma_\rho^{(1)}$, $\sigma_\vartheta^{(1)}$, $\tau_{\rho\vartheta}^{(1)}$ on the same circumference.

We introduce into consideration two new functions

$$\left.\begin{aligned} &\Phi^*(\zeta) = i\Phi(\zeta), \quad \Psi^*(\zeta) = i\Psi(\zeta), \\ &(\Phi_1(\zeta) = \Phi(\zeta) + \Phi_0(\zeta), \quad \Psi_1(\zeta) = \Psi(\zeta) + \Psi_0(\zeta)) \end{aligned}\right\} \qquad (41.4)$$

and designate the elastic state of plane strain characterized by them as the companion state. All quantities (potentials, stresses, etc.) pertaining to the companion state are starred.

Differentiate with respect to the argument ϑ the equality

$$\varphi(\sigma) - \omega(\sigma)\,\overline{\Phi(\sigma)} - \overline{\psi(\sigma)} = W(\sigma) \text{ on } \gamma, \qquad (41.5)$$

where

$$W(\sigma) = W_1(\sigma) - \left[\varphi_0(\sigma) - \omega(\sigma)\,\overline{\Phi_0(\sigma)} - \overline{\psi_0(\sigma)}\right];$$

equality (41.5) expresses the boundary condition of our problem. Noting that

$$\frac{d}{d\vartheta} = i\sigma\frac{\partial}{\partial\sigma} - i\bar{\sigma}\frac{\partial}{\partial\bar{\sigma}} \qquad (\sigma = e^{i\vartheta},\ \bar{\sigma} = e^{-i\vartheta}),$$

we find

$$\Phi(\sigma)-\overline{\Phi(\sigma)}+\frac{\bar{\sigma}^2}{\omega'(\sigma)}\left[\omega(\sigma)\overline{\Phi'(\sigma)}+\overline{\omega'(\sigma)\Psi(\sigma)}\right]=\Lambda(\sigma), \qquad (41.6)$$

where

$$\Lambda(\sigma)=\frac{1}{i\sigma\omega'(\sigma)}\frac{d}{d\vartheta}W(\sigma), \qquad (41.7)$$

or, with the notation (41.4),

$$\Phi^*(\sigma)+\overline{\Phi^*(\sigma)}-\frac{\sigma^2}{\overline{\omega'(\sigma)}}\left[\overline{\omega(\sigma)}\Phi^{*\prime}(\sigma)+\omega'(\sigma)\Psi(\sigma)\right]=-i\overline{\Lambda(\sigma)} \qquad (41.8)$$

Comparison of formulas (41.8) and (41.2) determines the external forces corresponding to the companion state as follows:

$$\tau_{\rho\vartheta}^*-i\sigma_\rho^*=\Lambda(\sigma) \text{ on } \gamma. \qquad (41.9)$$

Calculate the resultant of the external forces applied to the boundary of the body in the companion field. On the basis of formulas (39.38), (41.7), (41.9) we have

$$\int_L (X_n^*+iY_n^*)\,ds=\int_\gamma \Lambda(\sigma)\,\omega'(\sigma)\,d\sigma=\int_0^{2\pi}\frac{d}{d\vartheta}W(\sigma)\,d\vartheta=0,$$

and hence

$$X^*=Y^*=0, \qquad A^*=A^{*\prime}=0. \qquad (41.10)$$

Our nearest purpose is to determine the normal stress σ_ϑ^* on γ. But the problem of determining σ_ϑ^* under conditions (41.9) and (41.10) is nothing more than the problem of the preceding section, namely: find by relations (39.13) the function $\Omega(\sigma)$ from given values of the stress components σ_ρ and $\tau_{\rho\vartheta}$ on γ if, in addition, the given data include the first coefficients of the expansions in the neighbourhood of $\zeta=\infty$ of the functions $\Phi(\zeta)$ and $\Psi(\zeta)$ holomorphic in Σ^- and vanishing at infinity.

To find σ_ϑ^*, we thus have to solve the system of equations (39.34) with the right-hand sides defined by the data (41.9) and (41.10).

By solving this system, we find the unknown σ_ϑ^* in the form

$$\sigma_\vartheta^*+\sigma_\rho^*=\sum_{-\infty}^{\infty}a_k^*\sigma^k. \qquad (41.11)$$

It remains, finally, to determine the contour stresses in the basic field from the given values on γ of the functions u, v, σ_ρ^*, σ_ϑ^*, $\tau_{\rho\vartheta}^*$.

We remark first that formula (41.6) written in an alternative form implies the equality

$$\Phi(\sigma)-\overline{\Phi(\sigma)}+\frac{1}{2}(\sigma_\vartheta-\sigma_\rho)-i\tau_{\rho\vartheta}=\Lambda(\sigma),$$

whence

$$\sigma_\vartheta - \sigma_\rho = 2\Lambda_1 = 2\tau^*_{\rho\vartheta} \qquad (\Lambda(\sigma) = \Lambda_1(\sigma) + i\Lambda_2(\sigma)). \qquad (41.12)$$

Likewise, from the second formula of (41.1) which, with the use of the notation (41.4), we now rewrite as

$$\sigma_\vartheta - \sigma_\rho + 2i\tau_{\rho\vartheta} = -\frac{2i\sigma^2}{\overline{\omega'(\sigma)}}\left[\overline{\omega(\sigma)}\,\Phi^{*\prime}(\sigma) + \omega'(\sigma)\,\Psi^*(\sigma)\right] =$$
$$= 2\tau^*_{\rho\vartheta} - i(\sigma^*_\vartheta - \sigma^*_\rho)$$

it follows that

$$2\tau_{\rho\vartheta} = \sigma^*_\rho - \sigma^*_\vartheta. \qquad (41.13)$$

We now turn to the first of formulas (41.1)

$$2\left[\Phi^*(\sigma) + \overline{\Phi^*(\sigma)}\right] = \Omega^*(\sigma), \qquad \Omega^*(\sigma) = \sigma^*_\vartheta + \sigma^*_\rho.$$

From this, representing the function $\Phi^*(\zeta)$, as before, with the aid of the Schwarz integral, so that

$$2\Phi^*(\zeta) = -\frac{1}{2\pi i}\int\limits_\gamma \frac{\Omega^*(\sigma)\,d\sigma}{\sigma - \zeta} \qquad (\zeta \text{ in } \Sigma^-),$$

we find

$$2\Phi(\zeta) = \frac{1}{2\pi}\int\limits_\gamma \frac{\Omega^*(\sigma)\,d\sigma}{\sigma - \zeta} \qquad (\zeta \text{ in } \Sigma^-),$$

$$2\Phi(\sigma) = -\frac{i\Omega^*(\sigma)}{2} + \frac{1}{2\pi}\int\limits_\gamma \frac{\Omega^*(\tau)\,d\tau}{\tau - \sigma} \quad \text{on } \gamma.$$

The preceding formula, with (41.11), gives

$$\Omega(\sigma) = 4\operatorname{Re}\Phi(\sigma) = 2\operatorname{Re}\left\{\lim_{\zeta\to\sigma}\frac{1}{2\pi}\int\limits_\gamma \frac{\Omega^*(\tau)\,d\tau}{\tau - \zeta}\right\} =$$
$$= -2\operatorname{Re}\left\{i\sum_1^\infty \bar{a}^*_k\sigma^{-k}\right\}. \qquad (41.14)$$

On the basis of formulas (41.11) to (41.14) the unknown stresses along the contour of the hole are given in the following final form:

$$\left.\begin{aligned} \sigma_\vartheta &= \operatorname{Re}\Big\{i\sum_1^\infty a^*_k\sigma^k\Big\} + \Lambda_1(\sigma),\\ \sigma_\rho &= \operatorname{Re}\Big\{i\sum_1^\infty a^*_k\sigma^k\Big\} - \Lambda_1(\sigma),\\ -\tau_{\rho\vartheta} &= \operatorname{Re}\Big\{\sum_1^\infty a^*_k\sigma^k\Big\} + \Lambda_2(\sigma). \end{aligned}\right\} \qquad (41.15)$$

It will be recalled that a^*_k is the solution of an infinite system of the form of (39.34) and $\Lambda = \Lambda_1 + i\Lambda_2$ is defined by formula (41.7).

Of course, to the values (41.15) found above must be added the stresses corresponding to the given state at infinity.

Example. Infinite plane with a rigid elliptical inclusion*. Suppose that a rigid elliptical core inserted into a hole in an infinite elastic body is bonded to the surrounding medium along the contour of the hole, and the body is subjected at infinity to a simple tension perpendicular to the major axis of the ellipse. Let this major axis be coincident with the x axis and bisected by the point $z = 0$.

Then

$$\omega(\zeta) = R\left(\zeta + \frac{m}{\zeta}\right) \qquad (R > 0,\ 0 \leqslant m < 1), \tag{41.16}$$

$$W_1(\sigma) = 0 \text{ on } \gamma, \tag{41.17}$$

$$\Gamma = \frac{p}{4}, \qquad \Gamma' = \frac{p}{2}, \quad X = Y = 0, \tag{41.18}$$

where p is the intensity of tensile force at infinity.

The unknown functions φ_1, ψ_1 are represented as

$$\varphi_1(\zeta) = \varphi(\zeta) + \varphi_0(\zeta), \qquad \psi_1(\zeta) = \psi(\zeta) + \psi_0(\zeta),$$

where $\varphi(\zeta)$, $\psi(\zeta)$ are new unknown functions and

$$\left.\begin{aligned} \varphi_0(\zeta) &= \Gamma\omega(\zeta) = \frac{pR}{4}\left(\zeta + \frac{m}{\zeta}\right), \\ \psi_0(\zeta) &= \Gamma'\omega(\zeta) = \frac{pR}{2}\left(\zeta + \frac{m}{\zeta}\right). \end{aligned}\right\} \tag{41.19}$$

According to the preceding formulas, in our case

$$\Phi_0(\zeta) = \Gamma, \qquad \Psi_0(\zeta) = \Gamma' \tag{41.20}$$

and the right-hand side of equality (41.5) which expresses the boundary conditions of the problem presented here, with (41.17), becomes

$$W(\sigma) = \Gamma' R\left(\frac{1}{\sigma} + m\sigma\right). \tag{41.21}$$

From formulas (41.16) and (41.21) we find after elementary computations

$$\Lambda(\sigma) = \Gamma' \frac{m\sigma^2 - 1}{\sigma^2 - m}, \tag{41.22}$$

$$\overline{\sigma^2\omega'(\sigma)}\,\Theta^*(\sigma) = -2\overline{\sigma^2\omega'(\sigma)}\,(\sigma_\rho^* - i\tau_{\rho\vartheta}^*) = -2i\Gamma' R\frac{\sigma^2 - m}{\sigma^2}. \tag{41.23}$$

Further, on the basis of formulas (39.26) and (41.23)

$$F^*(\zeta) = -\frac{1}{2\pi i}\int\limits_{\gamma} \frac{\overline{\sigma^2\omega'(\sigma)}\,\Theta^*(\sigma)\,d\sigma}{\sigma - \zeta} = 2i\Gamma' R \qquad (\zeta \text{ in } \Sigma^+), \tag{41.24}$$

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* N. I. Muskhelishvili [93], Sec. 83a.

and hence for the given coefficients A_k of the companion problem we have

$$A_0^* = ipR, \qquad A_k^* = 0 \qquad (k \geqslant 1), \qquad A^{*\prime} = 0. \quad (41.25)$$

By solving system (39.34) with the right-hand sides (41.25), and calculating $\Omega^*(\sigma)$ by formula (41.11), we find*

$$\Omega^*(\sigma) = ip\left[\frac{\sigma^2}{1-m\sigma^2} - \frac{1}{\sigma^2-m}\right].$$

From this we have

$$i\sum_{1}^{\infty} a_k^* \sigma^k = -\frac{p\sigma^2}{1-m\sigma^2} \qquad (41.26)$$

and formulas (41.15) give, finally,

$$\left.\begin{aligned} \sigma_\vartheta &= -p\,\mathrm{Re}\left\{\frac{\sigma^2}{1-m\sigma^2}\right\} + \Lambda_1(\sigma), \\ -\sigma_\rho &= p\,\mathrm{Re}\left\{\frac{\sigma^2}{1-m\sigma^2}\right\} + \Lambda_1(\sigma), \\ -\tau_{\rho\vartheta} &= p\,\mathrm{Re}\left\{\frac{i\sigma^2}{1-m\sigma^2}\right\} + \Lambda_2(\sigma) \quad (\Lambda = \Lambda_1 + i\Lambda_2). \end{aligned}\right\} \qquad (41.27)$$

Calculate, finally, the stresses σ_ϑ^0, σ_ρ^0, $\tau_{\rho\vartheta}^0$ corresponding to potentials (41.20). On the basis of formulas (39.10) we easily obtain

$$\left.\begin{aligned} 2\sigma_\vartheta^0 &= p + p\,\mathrm{Re}\left\{\frac{\sigma^2-m}{1-m\sigma^2}\right\}, \\ 2\sigma_\rho^0 &= p - p\,\mathrm{Re}\left\{\frac{\sigma^2-m}{1-m\sigma^2}\right\}, \\ -2\tau_{\rho\vartheta}^0 &= p\,\mathrm{Re}\left\{\frac{i(\sigma^2-m)}{1-m\sigma^2}\right\}. \end{aligned}\right\} \qquad (41.28)$$

Forming now the sums, $\sigma_\vartheta + \sigma_\vartheta^0, \ldots,$ of the quantities defined by the preceding two sets of formulas, we find the unknown values of the stress components on γ as follows:

$$\sigma_\vartheta^{(1)} - i\tau_{\rho\vartheta}^{(1)} = \frac{p}{2}\,\frac{1-(m+2)\sigma^2}{1-m\sigma^2}, \qquad \sigma_\rho^{(1)} = \sigma_\vartheta^{(1)}. \qquad (41.29)$$

The tensile stress $\sigma_\vartheta^{(1)}$ at the pointed vertices of the elliptical core has the value

$$\sigma_\vartheta^{(1)} = -\frac{p}{2}\,\frac{1+m}{1-m} \qquad (\sigma = \pm 1). \qquad (41.30)$$

IV. APPROXIMATE SOLUTION OF PLANE PROBLEMS

In this part a method for the solution of plane problems, due to N. I. Muskhelishvili, is realized in a form convenient for calculations, a method using power series in conjunction with conformal mapping (N. I. Muskhelishvili [93], Sec. 63).

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* See Par. 4 of the preceding section.

It is remarkable that the proposed method of approximate solution does not assume a prior specification of a mapping function in any form. All necessary elements associated with conformal mapping are determined in the process of solution in numerical form.

To find the required mapping (in numerical form), use is made of the discrete analogue of the well-known Theodorsen-Garrick method in the theory of conformal maps (T. Theodorsen [131], T. Theodorsen and I. E. Garr ck [132]).

42. Method of Solution*

We shall consider the first fundamental problem for a finite or an infinite region S in the plane of the variable z bounded by a single closed contour L. According to the general consideration developed in Sec. 1, the boundary condition corresponding to the problem is

$$\varphi(\sigma)+\omega(\sigma)\frac{\overline{\varphi'(\sigma)}}{\overline{\omega'(\sigma)}}+\overline{\psi(\sigma)}=f(\sigma) \text{ on } \gamma, \tag{42.1}$$

where $\varphi(\sigma)$, $\psi(\sigma)$ are the boundary values of functions $\varphi(\zeta)$, $\psi(\zeta)$, holomorphic in the unit circle in the plane of the auxiliary variable ζ, $f(\sigma)$ is a given complex function on the boundary of the circle, and

$$z=\omega(\zeta)$$

denotes a function which effects a one-to-one conformal mapping of the circle $|\zeta|<1$ onto S.

1°. The idea of the above-mentioned method of N. I. Muskhelishvili is the following. Assume

$$\left.\begin{aligned}&\varphi(\zeta)=\sum_1^\infty a_k\zeta^k, &&\varphi'(\zeta)=\sum_1^\infty ka_k\zeta^{k-1},\\ &\psi(\zeta)=\sum_0^\infty a_k'\zeta^k &&(|\zeta|<1);\end{aligned}\right\} \tag{42.2}$$

$$\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}=\sum_{-\infty}^{\infty} b_k\sigma^k, \qquad f(\sigma)=\sum_{-\infty}^{\infty} A_k\sigma^k \text{ on } \gamma. \tag{42.3}$$

The above series are inserted, under certain assumptions as to their convergence on γ, in condition (42.1) and the coefficients of like powers of σ ($\sigma=e^{i\vartheta}$) are compared. For the determination of the unknown coefficients of the expansions, a_k, a_k', we then obtain infinite systems of linear algebraic equations

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* See the author's paper [50].

$$a_m+\sum_{1}^{\infty} k\bar{a}_k b_{m+k-1}=A_m \qquad (m\geqslant 1), \tag{42.4}$$

$$a'_m+\sum_{0}^{\infty} k a_k b_{-m+k-1}=A_{-m} \qquad (m\geqslant 0). \tag{42.5}$$

If the solution a_k of system (42.4) is found, all a'_k can be determined successively from system (42.5). The functions $\varphi(\zeta)$, $\psi(\zeta)$ constructed in this manner, together with the mapping function, completely define the unknown state of stress. In particular, the sum of the principal stresses is equal to

$$\sigma_x+\sigma_y=\operatorname{Re}\left\{\frac{\varphi'(\zeta)}{\omega'(\zeta)}\right\}. \tag{42.6}$$

With the mapping function $\omega(\zeta)$ given, the plane problem is thus reduced to the solution of system (42.4). It is proved that if (in the case of a finite region) the conditions of statics are fulfilled and if, further, the contour L and the right-hand side of condition (42.1) are sufficiently smooth, then system (42.4) does solve the original boundary value problem (42 1).*

Our nearest purpose is to determine the coefficients b_n.**

2°. For definiteness, suppose that the region S is finite and contains the origin of co-ordinates. The mapping function $\omega(\zeta)$ is subject to the normalization conditions

$$\omega(0)=0, \qquad \omega'(0)>0. \tag{42.7}$$

Let the boundary of the region S, a simple closed curve L, be given by the equation

$$\rho=\rho(\Phi), \tag{42.8}$$

where ρ is a sufficiently smooth function of the polar angle Φ $(0\leqslant \Phi\leqslant 2\pi)$. Denote by $\Phi(\vartheta)$ the relation established by the equation $t=\omega(\sigma)$ between the polar angles of the points of L and γ, $\Phi(\vartheta)=\arg\omega(e^{i\vartheta})$.

Consider the function

$$F(\zeta)=\ln\frac{\omega(\zeta)}{\zeta}.$$

On the basis of conditions (42.7) it is holomorphic for $|\zeta|<1$ and real at the origin. On the circumference γ we have

$$F(e^{i\vartheta})=\ln\rho[\Phi(\vartheta)]+i[\Phi(\vartheta)-\vartheta]. \tag{42.9}$$

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[*] For details see the papers by N. I. Muskhelishvili [88], and by D. I. Sherman [119].

[**] An alternative method of determining b_n based, too, on the finding of the mapping function $\omega(\zeta)$ in numerical form is given in the paper by V. L. Pisacane and L. E. Malvern [102].

The above equality means that the functions $\ln \rho\,[\Phi(\vartheta)]$ and $\Phi(\vartheta) - \vartheta$ are conjugate harmonic on the circumference. Hence,

$$\Phi(\vartheta) - \vartheta = -\frac{1}{2\pi}\int_0^{2\pi} \ln \rho\,[\Phi(\tau)] \cot\frac{\tau-\vartheta}{2}\,d\tau. \qquad (42.10)$$

The preceding relation represents a non-linear singular equation in $\Phi(\vartheta)$, first stated by T. Theodorsen [131]. When $\Phi(\vartheta)$ has been determined from Eq. (42.10), the boundary values $F(e^{i\vartheta})$ are found directly by formula (42.9).

To solve Eq. (42.10), which is frequently applied in practice to problems of flow of an incompressible ideal fluid, use is commonly made of the method of successive approximations.

The method of successive approximations as applied to Eq. (42.10) has been studied in detail by S. E. Warschawski [141]. Warschawski has established that, if the region S is reasonably close to a circle and its boundary is sufficiently smooth, the sequence $\Phi_n(\vartheta)$ determined from (42.10) by an iteration method (with the initial function $\Phi_0(\vartheta) = \vartheta$) converges, together with its derivative, to a unique solution of Eq. (42.10) which realizes a one-to-one conformal mapping of S onto the circle.

The work of A. Ostrowski [98] devoted to the same subject contains a somewhat more general result. In his other work ([99]), under the same conditions of closeness to a circle as those of S. E. Warschawski, A. Ostrowski establishes the convergence of the iteration method as applied to a discrete form of Eq. (42.10). This discrete analogue will be used below for the construction of numerical mapping.

3°. Denote by ν and β the angles that the outward normal to L at the point with polar angle Φ makes with the positive x direction and the radius vector respectively. The angle ν will be measured from the x axis, and β from the radius vector. Following the familiar convention for the measurement of these angles, we have

$$\nu(\Phi) = \Phi + \beta(\Phi) \qquad (-\pi/2 \leqslant \beta \leqslant \pi/2). \qquad (42.11)$$

The function $\ln \omega'(\zeta)$ is regular in the circle and real at $\zeta = 0$. For its boundary values we have

$$\ln \omega'(\sigma) = \ln|\omega'(\sigma)| + i \arg \omega'(\sigma). \qquad (42.12)$$

But

$$\arg \omega'(\sigma) = \nu - \vartheta = \beta + \Phi - \vartheta, \quad |\omega'(\sigma)| = \frac{ds}{d\vartheta} = \frac{ds}{d\Phi}\frac{d\Phi}{d\vartheta}, \qquad (42.13)$$

where s is the arc length of the contour L measured from its arbitrary point.

On the other hand,

$$\cos\beta = \rho\frac{d\Phi}{ds}, \qquad \frac{ds}{d\Phi} = \rho\sec\beta,$$

and hence

$$|\omega'(\sigma)| = \rho \sec \beta(\Phi)\, \Phi'(\vartheta).$$

Substituting the preceding expressions in relation (42.12), we obtain the equality

$$\ln \omega'(\sigma) = \ln \{\rho \sec \beta(\Phi)\, \Phi'(\vartheta)\} + i\, \{\Phi(\vartheta) + \beta(\Phi) - \vartheta\}$$

in which the pair of real functions on the right-hand side represent the contour values on the circumference of the regular conjugate harmonic functions in the circle. Taking into account the second condition of (42.7), we obtain, therefore,

$$\ln \{\rho(\Phi) \sec \beta(\Phi)\, \Phi'(\vartheta)\} = \\ = \frac{1}{2\pi} \int_0^{2\pi} \{\beta[\Phi(\tau)] + \Phi(\tau) - \tau\} \cot \frac{\tau - \vartheta}{2}\, d\tau + \ln \omega'(0).$$

Hence

$$|\omega'(\sigma)| = \exp\left[\frac{1}{2\pi} \int_0^{2\pi} \{\beta[\Phi(\tau)] + \Phi(\tau) - \tau\} \cot \frac{\tau - \vartheta}{2}\, d\tau + \ln \omega'(0)\right]. \tag{42.14}$$

This formula, together with relations (42.13), enables the contour values $\omega'(\sigma)$ to be expressed in terms of the solution of Eq. (42.10) and some known functions of the point on the contour L, viz. elements of the same contour. We have

$$\omega'(\sigma) = \exp\left[\frac{1}{2\pi} \int_0^{2\pi} \{\beta[\Phi(\tau)] + \Phi(\tau) - \tau\} \cot \frac{\tau - \vartheta}{2}\, d\tau + \right. \\ \left. + \ln \omega'(0) + i\{\beta[\Phi(\vartheta)] + \Phi(\vartheta) - \vartheta\}\right]. \tag{42.15}$$

On the basis of formula (42.9) we write similarly to the above

$$\ln \rho[\Phi(\vartheta)] = \frac{1}{2\pi} \int_0^{2\pi} \{\Phi(\tau) - \tau\} \cot \frac{\tau - \vartheta}{2}\, d\tau + \ln \omega'(0).$$

By using this equality, the contour values of the mapping function can be given the form

$$\omega(\sigma) = \rho[\Phi(\vartheta)]\, e^{i\Phi(\vartheta)} = \\ = \exp\left[\frac{1}{2\pi} \int_0^{2\pi} \{\Phi(\tau) - \tau\} \cot \frac{\tau - \vartheta}{2}\, d\tau + \ln \omega'(0) + i\Phi(\vartheta)\right]. \tag{42.16}$$

Equalities (42.15) and (42.16) allow the required coefficient of the function $\varphi'(\sigma)$ in condition (42.1) to be represented as

$$\frac{\omega(\sigma)}{\omega'(\sigma)}=\exp\left[-\frac{1}{2\pi}\int_0^{2\pi}\{\nu[\Phi(\tau)]-\Phi(\tau)\}\cot\frac{\tau-\vartheta}{2}\,d\tau+\right.$$

$$\left.+i\{\nu[\Phi(\vartheta)]+\Phi(\vartheta)-\vartheta\}\right]. \qquad (42.17)$$

4°. The method of successive approximations for the solution of Eq. (42.10) is conveniently realized in discrete form owing to the high accuracy of simple quadrature formulas for the improper integral

$$g(\vartheta)=\frac{1}{2\pi}\int_0^{2\pi}f(\tau)\cot\frac{\tau-\vartheta}{2}\,d\tau. \qquad (42.18)$$

The interpolation nodes on $[0, 2\pi]$ are taken to be the points

$$\vartheta_m=\frac{m\pi}{N} \qquad (m=0, 1, \ldots, 2N-1),$$

where N is an arbitrary natural number. Then the following quadrature formula is valid (H. Serbin [113]; see also A. A. Korneichuk [61])

$$g_m=\sum_{k=0}^{2N-1}\alpha_{k-m}f_k \qquad (m=0, 1, \ldots, 2N-1), \qquad (42.19)$$

where

$$\alpha_{k-m}=\begin{cases}0 & \text{if } |k-m| \text{ is even},\\ \frac{1}{N}\cot\frac{\vartheta_k-\vartheta_m}{2} & \text{if } |k-m| \text{ is odd}.\end{cases} \qquad (42.20)$$

Formula (42.19) is accurate whenever $f(\vartheta)$ is a trigonometric polynomial of degree not higher than $N-1$.*

According to formula (42.20), relations (42.19) fall into two groups, namely

$$g_{2m}=\sum_{k=0}^{N-1}\alpha_{2(k-m)+1}f_{2k+1},$$

$$g_{2m+1}=\sum_{k=0}^{N-1}\alpha_{2(k-m)-1}f_{2k} \qquad (m=0, 1, \ldots, N-1),$$

and in consequence the application of the iteration method to the system with matrix (α_{k-m}) appears to be extremely convenient.

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* The question of convergence of the quadrature formula (42.19) for various classes of functions $f(\vartheta)$ has been studied in the above-mentioned work of A. A. Korneichuk [61].

On the basis of formula (42.19) the discrete form of Eq. (42.10) is the following:

$$\Phi_m = \vartheta_m - \sum_{k=0}^{2N-1} \alpha_{k-m} \ln \rho\,[\Phi_k] \qquad (m=0,\ 1,\ \ldots,\ 2N-1), \qquad (42.21)$$

$$\Phi_m = \Phi(\vartheta_m), \qquad \rho\,[\Phi_m] = \rho\,[\Phi(\vartheta_m)].$$

For regions which are fairly close to a circle the iteration must be begun with the values

$$\Phi_m^{(0)} = \vartheta_m \qquad (m=0,\ 1,\ \ldots,\ 2N-1).$$

Assume

$$\frac{\omega(\sigma)}{\omega'(\sigma)} = F_1(\vartheta) + iF_2(\vartheta), \qquad (42.22)$$

where

$$F_1(\vartheta) = \Lambda(\vartheta)\cos\Omega(\vartheta), \qquad F_2(\vartheta) = \Lambda(\vartheta)\sin\Omega(\vartheta),$$

$$\Lambda(\vartheta) = \exp\left[-\frac{1}{2\pi}\int_0^{2\pi}\{\nu[\Phi(\tau)] - \Phi(\tau)\}\cot\frac{\tau-\vartheta}{2}\,d\tau\right]$$

and

$$\Omega(\vartheta) = \nu\,[\Phi(\vartheta)] + \Phi(\vartheta) - \vartheta. \qquad (42.23)$$

To calculate the coefficients b_m defined by the equalities

$$b_m = \frac{1}{2\pi}\int_0^{2\pi} \frac{\omega(\sigma)}{\omega'(\sigma)} \exp(-im\vartheta)\,d\vartheta \qquad (m=0,\ \pm 1,\ \pm 2,\ \ldots), \qquad (42.24)$$

we use the rectangular formula giving the exact value of the integral for trigonometric polynomials of degree not higher than $2N-1$

$$\frac{1}{\pi}\int_0^{2\pi} F(\vartheta)\exp(-im\vartheta)\,d\vartheta = \frac{1}{N}\sum_{k=0}^{2N-1} F(\vartheta_k)\exp(-im\vartheta_k). \qquad (42.25)$$

The result is

$$b_m = \frac{1}{2N}\sum_{k=0}^{2N-1}\{F_1(\vartheta_k) + iF_2(\vartheta_k)\}\exp(-im\vartheta_k) \qquad (m=0,\ \pm 1,\ \ldots), \qquad (42.26)$$

where, by formulas (42.19) and (42.23),

$$F_1(\vartheta_m) + iF_2(\vartheta_m) = \exp\left[-\sum_{k=0}^{2N-1}\alpha_{k-m}(\nu_k - \Phi_k) + i(\nu_m + \Phi_m - \vartheta_m)\right]. \qquad (42.27)$$

Ultimately the sequence of operations for the solution of the boundary value problem (42.1) is as follows.

First solve the non-linear equation (42.21) and find the values of the function $\Phi(\vartheta)$ at the nodes ϑ_m. From given Φ_m for a given angle $\nu(\Phi)$ calculate the requisite number of coefficients b_m. Thereafter, by solving the shortened system (42.4) with the requirement that the resultant couple of the external forces is zero, find the first coefficients a_m in the expansion of the function $\varphi(\zeta)$ and then, from system (42.5), the first coefficients of the function $\psi(\zeta)$.

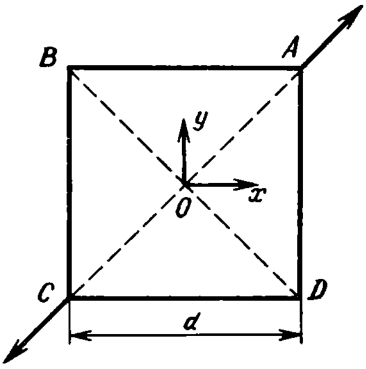

Fig. 8

The boundary values on the network of the mapping function are calculated by formula (42.16); they, together with expression (42.27), determine the discrete values $\omega'(\sigma)$ as

$$\omega'(\exp(i\vartheta_m)) = \frac{\rho_m \exp(-i\Phi_m)}{F_1(\vartheta_m) - iF_2(\vartheta_m)}. \tag{42.28}$$

To determine the same functions inside the circle, use should be made of Cauchy's formula

$$\omega(\zeta) = \frac{1}{2\pi i} \int\limits_{\nu_\gamma} \frac{\omega(\sigma)\,d\sigma}{\sigma-\zeta},$$

$$\omega'(\zeta) = \frac{1}{2\pi i} \int\limits_{\nu_\gamma} \frac{\omega'(\sigma)\,d\sigma}{\sigma-\zeta} \tag{42.29}$$

with a subsequent replacement of the integrals by finite sums with one or another method of approximate integration. On the basis of formulas (42.25) and (42.29) we have the following approximate formulas, strictly valid inside the circle:

$$\omega(\zeta) = \frac{1}{2N} \sum_{k=0}^{2N-1} \frac{\rho_k \exp[i(\Phi_k + \vartheta_k)]}{\exp(i\vartheta_k) - \zeta},$$

$$\omega'(\zeta) = \frac{1}{2N} \sum_{k=0}^{2N-1} \frac{\rho_k \exp[-i(\Phi_k - \vartheta_k)]}{[F_1(\vartheta_k) - iF_2(\vartheta_k)](\exp(i\vartheta_k) - \zeta)}.$$

The functions φ, φ', ψ, ω, ω' completely define the displacement field as well as the distribution of the sum of the principal stresses in an elastic body. To calculate all stress components separately, it is necessary yet to know a reasonably accurate expression for the second derivative $\omega''(\zeta)$.

Example. Consider an elastic square stretched along a diagonal by two equal and opposite concentrated forces applied at opposite vertices. The side d of the square is taken to be 2 and its centre is placed at the origin (Fig. 8).

Denote by A, B, C, D successively the vertices of the square corresponding to angles of $\pi/4$, $3\pi/4$, $5\pi/4$, $7\pi/4$.

For definiteness, it is supposed that the concentrated forces of unit magnitude are applied at points A and C along the line AC.

In the case under consideration, for the functions ρ and ν we have*

$$\begin{array}{lll}
\rho(\Phi)=\dfrac{1}{\cos\Phi}, & \nu(\Phi)=0, & 0\leqslant\Phi\leqslant\pi/4,\\
\rho(\Phi)=\dfrac{1}{\sin\Phi}, & \nu(\Phi)=\pi/2, & \pi/4\leqslant\Phi\leqslant3\pi/4,\\
\rho(\Phi)=-\dfrac{1}{\cos\Phi}, & \nu(\Phi)=\pi, & 3\pi/4\leqslant\Phi\leqslant5\pi/4,\\
\rho(\Phi)=-\dfrac{1}{\sin\Phi}, & \nu(\Phi)=3\pi/2, & 5\pi/4\leqslant\Phi\leqslant7\pi/4,\\
\rho(\Phi)=\dfrac{1}{\cos\Phi}, & \nu(\Phi)=2\pi, & 7\pi/4\leqslant\Phi\leqslant2\pi.
\end{array}$$

It should be borne in mind that for legitimacy of the reasoning used above it is necessary to require the boundary of the region S to be reasonably smooth, e.g. that the contour represent Lyapunov's curve. When there are corners in L, the function $\omega'(\sigma)$, as is known from the theory of conformal maps [this is readily seen, too, from expression (42.14)], becomes infinite or zero at points corresponding to these corners. For this reason, in carrying out our algorithm we shall have to deal with large and small quantities near the above-mentioned points of the circumference, a fact which has an adverse effect on the computational work. But even when there are corners present, one can attempt to apply the scheme directly if the nodal points of the chosen network are isolated from the "dangerous" points. Then, however, we shall have a contour with rounded corners. Besides, the complex potentials φ and ψ will possess (implicit) singularities on the circumference which are characteristic of contour concentrated loads. This circumstance, of course, also presents some inconvenience in obtaining an acceptable solution to the problem.

In order to solve system (42.4) we must, in the first place, know the Fourier coefficients of the function $f=f_1+if_2$ with positive subscripts. These coefficients are most easily determined by expanding the Cauchy-type integral

$$F(\zeta)=\frac{1}{2\pi i}\int\limits_{\gamma}\frac{f(\sigma)\,d\sigma}{\sigma-\zeta} \qquad (42.30)$$

in the circle in a power series.

It is well to recall the formula

$$f_1+if_2=i\int\limits_0^s(X_n+iY_n)\,ds. \qquad (42.31)$$

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* For the function $\nu(\Phi)$ the equality signs in the conditions defining Φ must be omitted.

where X_n, Y_n are the components of the given stress vector applied at the boundary of the region S and the integral is taken with respect to the arc length s of the boundary contour. According to the relation $t = \omega(\sigma)$, the arc length s is a certain function of the arc length of the circumference γ, or, what is equivalent, of the point $\sigma = e^{i\vartheta}$. The right-hand side of formula (42.31) may therefore be considered as a given function of σ. For our example we have

$$f_1 + if_2 = \exp(i\pi/4) \text{ on } ABC, \qquad f_1 + if_2 = 0 \text{ on } CDA.$$

Integral (42.30) now takes the form

$$F(\zeta) = \frac{\exp(i\pi/4)}{2\pi} \int_{\gamma_0} \frac{d\sigma}{\sigma - \zeta}, \tag{42.32}$$

where γ_0 is the image of ABC in the mapping $t = \omega(\sigma)$.

We have

$$F(\zeta) = \frac{\exp(i\pi/4)}{2\pi} \ln \frac{c-\zeta}{a-\zeta},$$

where $a = e^{i\alpha}$ and c are the images of the points A and C, respectively, on the circumference γ.

On expanding this function in a series for $|\zeta| < 1$ and setting $c = -a$ in view of the symmetry, we find

$$F(\zeta) = \frac{\exp(i\pi/4)}{\pi} \left[\frac{\pi i}{2} + \frac{\zeta}{a} + \frac{\zeta^3}{3a^3} + \dots \right].$$

Consequently,

$$A_0 = \frac{i \exp(i\pi/4)}{4}, \qquad A_{2k} = 0, \qquad A_{2k-1} = \frac{\exp i[\pi/4 - (2k-1)\alpha]}{\pi(2k-1)}.$$

When $\alpha = \pi/4$, for the coefficients A_{2k-1} we have

$$A_{2k-1} = \frac{i^{1-k}}{\pi(2k-1)} \qquad (k = 1, 2, \dots). \tag{42.33}$$

If, in accordance with the above, we now take $N = 4k + 2$, where k is an integer, the points of the circumference $\vartheta_k = (2k-1)\pi/4$ $(k = 1, 2, 3, 4)$ will be at the middle of the length between two adjacent nodes of the given network. Table 1 gives the values of the function $\Phi(\vartheta)$ found from (42.21) by the iteration method to within 0.1 per cent with $N = 14$.

Table 1

$\vartheta_m = m\pi/14$

ϑ	ϑ_0	ϑ_1	ϑ_2	ϑ_3	ϑ_4	ϑ_5	ϑ_6	ϑ_7
Φ	0.0002	0.1753	0.3456	0.5941	0.9767	1.2252	1.3954	1.5710

As seen from Table 1, the polar angles ϑ of the points of the circumference are only slightly affected by the transformation of the circle, i.e., to a close approximation, it may be taken that

$$\Phi(\vartheta) = \vartheta.$$

Consequently, the assumptions as to the agreement between the boundary points of the circumference and its image, made above in the calculation of coefficients (42.33), are justified to a desired accuracy.

In the first equality of (42.3) the coefficients b_m, $m = 0, \pm 1, \ldots, \pm 32$, were calculated and the truncated system (42.4) was considered with $m = 16$, $k = 1, 2, \ldots, 16$. The resultant system for a_m consists of 31 linear equations with the same number of unknowns. Without going into details, we give, in Table 2, the discrete values calculated in this manner for the tensile stress σ_y along the segment DA.

Table 2

$\vartheta_m = m\pi/16$

ϑ	ϑ_{29}	ϑ_{30}	ϑ_{31}	ϑ_0	ϑ_1	ϑ_2	ϑ_3
σ_y	1.9747	2.2030	3.5638	3.6514	3.9636	4.5741	10.9568

It should be remembered that in the case under consideration the series for $\omega(\zeta)$ in the circle does not converge at all at the corners, and the first series of (42.3) converges very slowly. Besides, the series (42.2) diverge at the points of application of the concentrated loads which coincide in our example with the angular points. For this reason, the coefficients b_m and a_m decrease rather slowly, a fact which is confirmed by the calculations.

Nevertheless, Table 2 gives a useful qualitative picture of the contour stress. In a quantitative respect, however, the present order of approximation is possibly not quite satisfactory.

We remark finally that the same example has been previously treated by the complex variable method in a paper of C. A. M. Gray [31], where polynomial approximations have been used for mapping the square.

43. Continuation*

The system of equations (42.4) can be obtained from the functional equation of N. I. Muskhelishvili ([93], Sec. 78)

$$\varphi(\zeta) + \frac{1}{2\pi i}\int\limits_{\gamma} \frac{\omega(\sigma)\,\overline{\varphi'(\sigma)}}{\overline{\omega'(\sigma)}}\,\frac{d\sigma}{\sigma-\zeta} = \frac{1}{2\pi i}\int\limits_{\gamma} \frac{f(\sigma)\,d\sigma}{\sigma-\zeta} + \text{constant} \quad (|\zeta| < 1) \tag{43.1}$$

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* The present section reproduces the content of the paper by O. E. Laliashvili [69] in abridged form.

by replacing the functions occurring in it by the corresponding series (42.2) and (42.3). In the case of angles in the contour smaller than π and, also, when we are mainly concerned with the distribution of the sum of the normal stresses, $\sigma_x + \sigma_y$, it is more expeditious to proceed from another system of linear equations which will presently be indicated (see C. A. Gray [31]).

Assume for $|\zeta| < 1$

$$\frac{\varphi'(\zeta)}{\omega'(\zeta)} = \sum_0^\infty u_k \zeta^k, \tag{43.2}$$

$$\omega(\zeta) = \sum_1^\infty c_k \zeta^k \qquad (c_1 \neq 0). \tag{43.3}$$

The power series for $\varphi'(\zeta)$ is obviously obtained from relation (43.2), and the Fourier series for the unknown density function of the Cauchy-type integral on the left-hand side of (43.1) by the multiplication of series (43.2) and (43.3), after replacing ζ by σ in the former and passing to the conjugate values. These series, together with the second series of (42.3), are inserted in the equation

$$\varphi'(\zeta) + \frac{1}{2\pi i}\frac{d}{d\zeta}\int_\gamma \frac{\omega(\sigma)\,\overline{\varphi'(\sigma)}}{\overline{\omega'(\sigma)}}\frac{d\sigma}{\sigma-\zeta} = \frac{1}{2\pi i}\frac{d}{d\zeta}\int_\gamma \frac{f(\sigma)\,d\sigma}{\sigma-\zeta} \tag{43.4}$$

which is obtained from Eq. (43.1) by differentiation with respect to ζ; the differentiation of the Cauchy-type integrals is performed after representing them as power series.

On comparing the coefficients of ζ^m ($m = 0, 1, \ldots$), we obtain, after some simplification, the required system for the determination of the unknowns u_k

$$\sum_{k=0}^{m-1} [(m-k)\, c_{m-k} u_k + m c_{m+k} \bar{u}_k] + \\ + \sum_{k=0}^{\infty} m c_{2m+k} \bar{u}_{m+k} = m A_m \qquad (m = 1, 2, \ldots). \tag{43.5}$$

The coefficients

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} \omega(\sigma)\, e^{-in\vartheta}\, d\vartheta \qquad (n \geqslant 1) \tag{43.6}$$

will be determined by the method developed in the preceding section. For this we need to solve, as before, the non-linear system (42.21), make use of formula (42.16) which, with the equality

$$\ln \omega'(0) = \frac{1}{2\pi}\int_0^{2\pi} \ln \rho\,[\Phi(\vartheta)]\, d\vartheta,$$

we now represent in the form

$$\omega(\sigma) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi}[\Phi(\tau)-\tau]\cot\frac{\tau-\vartheta}{2}\,d\tau + \right.$$
$$\left. + \frac{1}{2\pi}\int_0^{2\pi}\ln\rho\,[\Phi(\tau)]\,d\tau + i\Phi(\vartheta)\right\}, \qquad (43.7)$$

and, finally, after finding discrete values of this function, calculate coefficients (43.6) by the application of the quadrature formula (42.25).

When the solution u_k of system (43.5) has been found, and hence the function

$$\Phi(\zeta) = \frac{\varphi'(\zeta)}{\omega'(\zeta)}$$

has been determined, the other unknown function of the plane problem, $\psi'(\zeta)$, is determined directly from the boundary condition (42.1) which can be taken in the alternative form [see formula (1.18)]

$$\Phi(\sigma) + \overline{\Phi(\sigma)} - \frac{\sigma^2}{\overline{\omega'(\sigma)}}\left[\overline{\omega(\sigma)}\,\Phi'(\sigma) + \psi'(\sigma)\right] = \sigma_\rho - i\tau_{\rho\vartheta} \text{ on } \gamma. \qquad (43.8)$$

The right-hand side of this equality must be considered as a given function of the point σ on γ; it is easily expressed in terms of $f(\sigma)$.

Example. By way of illustration of the solution scheme, we have considered, as an example, an elastic elliptical disk stretched along the major axis of the ellipse by two equal and oppositely directed concentrated forces applied to its vertices.

To facilitate the finding of a mapping function, we assume, to a high accuracy, that the disk is close to a circular one, and set $a = 1.5$, $b = 1$, where a and b are the major and minor semi-axes of the ellipse.

The non-linear system (42.21) has been solved by the iteration method* with $N = 16$. The values of Φ_m so obtained are given in Table 1.**

As seen from this table, when the ellipse is mapped the rotation of its points is almost entirely absent, and it may therefore be assumed, to a good approximation, that the image of the upper half of the ellipse is the upper semi-circumference in the ζ plane.

Thereafter, in exactly the same way as in the example of the preceding section we can calculate the coefficients A_m appearing on the

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* All computations in both this and the preceding example have been carried out using a БЭСМ-2 high-speed electronic computer.

** The calculations have been performed to 10^{-7}. All tables in the present section, except Table 2. give round-number figures.

Table 1

$\vartheta_m = m\pi/16$

m	Φ_m	m	Φ_m	m	Φ_m
0	0	11	2.0617	22	4.3924
1	0.2437	12	2.2455	23	4.5546
2	0.4778	13	2.4457	24	4.7124
3	0.6958	14	2.6638	25	4.8702
4	0.8965	15	2.8979	26	5.0324
5	1.0799	16	3.1416	27	5.2033
6	1.2508	17	3.3853	28	5.3871
7	1.4130	18	3.6193	29	5.5873
8	1.5708	19	3.8374	30	5.8054
9	1.7286	20	4.0376	31	6.0395
10	1.8908	21	4.2215		

right-hand side of formula (43.5). It is easily verified that in our case*

$$A_m = -\frac{1}{\pi m} \qquad (m \geqslant 1). \tag{43.9}$$

The next step in the proposed process of solution is the determination of coefficients (43.6). The calculations show that, to a sufficient approximation,

$$\alpha_{2k} = \beta_k = 0 \qquad (c_k = \alpha_k + i\beta_k, \quad k \geqslant 1). \tag{43.10}$$

As regards the numbers α_{2k+1}, they are different from zero, but the sequence $\{\alpha_{2k+1}\}$ decreases fairly rapidly. The numerical values of these coefficients are presented in the following table:

Table 2

k	α_{2k+1}	k	α_{2k+1}	k	α_{2k+1}
0	0.88898	4	0.0001355	8	0.00000002
1	0.09886	5	0.0000151	9	0
2	0.01097	6	0.0000017	10	
3	0.00122	7	0.0000002		

The data of Table 2 have been used to solve system (43.5), with the right-hand sides (43.9), in which all equations and unknowns u_m beginning with $m = 17$ are rejected. This gives a finite system of 32 real equations, with the same number of unknowns, for the real and imaginary parts of the first coefficients of u_k. Using the approximate solution u_k of system (43.5) so found, values of the sum

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* The magnitude of the concentrated force is taken to be unity.

$$Z(\xi, \eta) = \sigma_\rho + \sigma_\vartheta \tag{43.11}$$

have been calculated at equally spaced points of radii of the circle $|\zeta| < 1$ lying on the co-ordinate axes, the spacing being $h = 0.1$. These discrete values

$$Z(kh, 0) = Z_1^{(k)}, \qquad Z(0, kh) = Z_2^{(k)}$$

are presented below in tabular form.

Table 3 reproduces the distribution of invariant (43.11) with fair accuracy.

Table 3

k	0	1	2	3	4	5	6	7	8	9
$Z_1^{(k)}$	0.6500	0.6969	0.7511	0.8400	0.9801	1.2065	1.5985	2.3402	3.9577	7.7289
$Z_2^{(k)}$	0.6560	0.6466	0.6194	0.5777	0.5259	0.4687	0.4101	0.3518	0.2849	0.1526

44. To Conformal Mapping of Singly Connected Regions

Conformal mapping is essentially used in the solution of many problems of an applied nature. It is often very important, therefore, to know how to construct mapping functions in a form suitable for practical purposes.

At present a variety of methods are available for the construction of conformal maps.* The most popular among these are those based on the application of both linear and non-linear integral equations. We have already encountered one of these methods in the preceding sections.

Below we shall consider a method of successive approximations for mapping a unit circle onto a given (singly connected) region.**

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* See collected works: "Construction and Applications of Conformal Maps", Nat. Bur. Standards, Appl. Math. Ser., 18, 1952; "Experiment in the Computation of Conformal Maps", Nat. Bur. Standards, Appl. Math. Ser., 42, 1955; "Numerical Analysis", Proc. Sympos. Appl. Math., 6, 1956; see also the monographs by L. V. Kantorovich and V. I. Krylov [57] and by M. A. Lavrent'ev and B. V. Shabat [71].

** See the author's paper [47].

Suppose that a singly connected, and (for definiteness) finite, region S is given in the plane of the variable $z = x + iy$, which is bounded by a closed Lyapunov curve L and contains the origin of co-ordinates. Let the curve L be given by the polar equation

$$\rho = \rho\,(\Phi) \qquad (0 \leqslant \Phi \leqslant 2\pi),$$

where the function $\rho\,(\Phi)$ is continuously differentiable. The unknown function $z = \omega\,(\zeta)$ which maps the circle $|\,\zeta\,| < 1$ onto S conformally and in a one-to-one manner will be subjected to the ordinary normalization conditions

$$\omega\,(0) = 0, \qquad \omega'\,(0) > 0. \tag{44.1}$$

The proposed mapping method implies the availability of some initial mapping $z = \omega_0\,(\zeta)$ subject to the same normalization conditions

$$\omega_0\,(0) = 0, \qquad \omega_0'\,(0) > 0 \tag{44.2}$$

and close in a sense to the unknown one. In many cases of particular importance for applications this "close mapping" is not difficult to prescribe in a simple analytic form.

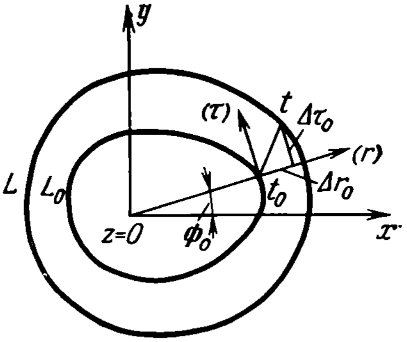

Fig. 9

Let us calculate the difference $\omega\,(\zeta) - \omega_0\,(\zeta)$.

Denote by L_0 the image of the circumference γ ($|\,\sigma\,| = 1$) in the initial mapping $\varphi_0\,(\zeta)$ (Fig. 9). Let t and t_0 be points on L and L_0, respectively, corresponding to a certain point σ on γ, $\sigma = e^{i\vartheta}$. Draw through the point t_0 two axes: an (r) axis which is a prolongation of the radius-vector at the point t_0, and a (τ) axis perpendicular to the first. The projections of the vector $\overline{t_0t}$ on (r) and (τ) are denoted by Δr_0 and $\Delta\tau_0$, respectively.

We must obviously have (see Fig. 9)

$$t - t_0 = [\Delta r_0 + i\Delta\tau_0]\, e^{i\Phi_0}, \qquad \Phi_0 = \arg t_0. \tag{44.3}$$

But

$$e^{i\Phi_0} = \frac{\omega_0\,(\sigma)}{|\,\omega_0\,(\sigma)\,|},$$

and hence equality (44.3) may be represented as

$$\frac{\omega\,(\sigma) - \omega_0\,(\sigma)}{\omega_0\,(\sigma)} = \frac{\Delta r_0 + i\Delta\tau_0}{|\,\omega_0\,(\sigma)\,|}. \tag{44.4}$$

Noting that, by formulas (44.1) and (44.2), the ratio $\frac{\omega(\zeta) - \omega_0(\zeta)}{\omega_0(\zeta)}$

is regular in the circle and real at $\zeta = 0$, we have from relation (44.4) on the basis of the Schwarz formula

$$\frac{\omega(\zeta)-\omega_0(\zeta)}{\omega_0(\zeta)} = \frac{1}{2\pi i}\int\limits_{\gamma} \frac{\Delta r_0(\vartheta)}{|\omega_0(\sigma)|}\,\frac{\sigma+\zeta}{\sigma-\zeta}\,\frac{d\sigma}{\sigma}. \tag{44.5}$$

The quantity $\Delta r_0(\vartheta)$ is not known to us. By replacing this quantity under the integral in (44.5) by a close value $\delta r_0(\vartheta) = \rho(\Phi_0) - |\omega_0(\sigma)|$, we obtain an approximate formula:

$$\omega(\zeta) = \omega_0(\zeta) + \frac{\omega_0(\zeta)}{2\pi i}\int\limits_{\gamma} \frac{\delta r_0(\vartheta)}{|\omega_0(\sigma)|}\,\frac{\sigma+\zeta}{\sigma-\zeta}\,\frac{d\sigma}{\sigma}. \tag{44.6}$$

For further refinement of the mapping function we shall take the right-hand side of (44.6) as a first approximation and find successive approximations by the same formula. Thus, for the $(n+1)$th approximation we have

$$\omega_{n+1}(\zeta) = \omega_n(\zeta) + \frac{\omega_n(\zeta)}{2\pi i}\int\limits_{\gamma} \frac{\delta r_n(\vartheta)}{|\omega_n(\sigma)|}\,\frac{\sigma+\zeta}{\sigma-\zeta}\,\frac{d\sigma}{\sigma}, \tag{44.7}$$

$$\delta r_n(\vartheta) = \rho\,[\Phi_n(\vartheta)] - |\,\omega_n(\sigma)\,| \qquad (n = 0, 1, \ldots). \tag{44.8}$$

Inserting expression (44.8) in formula (44.7), we can represent the latter in the following simpler form:

$$\omega_{n+1}(\zeta) = \frac{\omega_n(\zeta)}{2\pi i}\int\limits_{\gamma} \frac{\rho\,[\Phi_n(\vartheta)]}{|\omega_n(\sigma)|}\,\frac{\sigma+\zeta}{\sigma-\zeta}\,\frac{d\sigma}{\sigma} \qquad (n = 0, 1, \ldots). \tag{44.9}$$

For a practical realization of the algorithm, instead of expression (44.9) it is advisable to use a formula obtained from it by a limiting process as $\zeta \to \sigma_0$, $\sigma_0 = e^{i\vartheta_0}$. This formula, having the form

$$\omega_{n+1}(\sigma_0) = \omega_n(\sigma_0)\left\{\frac{\rho\,[\Phi_n(\vartheta_0)]}{|\,\omega_n(\sigma_0)\,|} + \frac{1}{2\pi i}\int\limits_0^{2\pi} \frac{\rho\,[\Phi_n(\vartheta)]}{|\,\omega_n(\sigma)\,|}\cot\frac{\vartheta-\vartheta_0}{2}\,d\vartheta\right\}, \tag{44.10}$$

is very convenient for numerical computation.

The accuracy of the first approximations determined by formula (44.10) depends to a considerable extent on the choice of the initial mapping $\omega_0(\zeta)$. For regions close to a unit circle it is natural to take $\omega_0(\zeta) = \zeta$. Note that in this case formula (44.6) is identical with the well-known close mapping formula available in the literature (see, for example, M. A. Lavrent'ev and B. V. Shabat [71], Chap. IV, Sec. 2, Par. 63).

Without touching upon the general theoretical aspects relating to the justification of the method,* we shall only illustrate the computational scheme in the particular case when the region S represents the inside of an ellipse given by the equation

$$\rho(\Phi)=\frac{a}{\sqrt{1+k^2\sin^2\Phi}}\qquad\left(k^2=\frac{a^2-b^2}{b^2}\right),\tag{44.11}$$

where a and b are the semi-axes of the ellipse.

In the case under consideration it appears advisable to choose the initial mapping $\omega_0(\zeta)$ in the form

$$\omega_0(\zeta)=\zeta e^{\alpha\zeta^2+\beta},\tag{44.12}$$

where α, β are constants determined from the condition of coincidence of the exact and approximate curves at the points $\zeta = 1$ and $\zeta = i$ in terms of a and b by the equalities

$$\alpha=\ln\sqrt{\frac{a}{b}},\qquad \beta=\ln\sqrt{ab}\left(\frac{a}{b}<e\right).\tag{44.13}$$

To find the first approximation by formula (44.10), we need to know values of $\rho[\Phi_0(\vartheta)]$. In order to calculate this function, it is sufficient to determine $\Phi_0(\vartheta) = \arg\omega_0(\zeta)$ from (44.12) and substitute the argument found into the right side of relation (44.11) for Φ. The result is

$$G(\vartheta)=\frac{\rho[\Phi_0(\vartheta)]}{|\omega_0(\sigma)|}=\frac{ae^{-\alpha\cos 2\vartheta-\beta}}{\sqrt{1+k^2\sin^2[\vartheta+\alpha\sin 2\vartheta]}}.\tag{44.14}$$

By assigning now in the interval $(0, \pi)$ n nodes

$$\vartheta_\nu=\frac{2\nu-1}{2n}\pi\qquad(\nu=1, 2, \ldots, n),$$

we construct an even trigonometric polynomial $G_{n-1}(\vartheta)=\sum\limits_{\nu=0}^{n-1}a_\nu\cos\nu\vartheta$ which interpolates $G(\vartheta)$ at the given nodes. According to formula (44.14), this polynomial will not contain terms with odd subscripts. Denoting further by $G^*_{n-1}(\vartheta)$ a trigonometric polynomial conjugate to $G_{n-1}(\vartheta)$, we find the first approximation to the unknown mapping

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* It is of much interest to investigate the convergence of process (44.10) that is, the convergence of the method of successive approximations for the non-linear functional equation

$$\omega(\sigma_0)=\omega(\sigma_0)\left\{\frac{\rho[\arg\omega(\sigma_0)]}{|\omega(\sigma_0)|}+\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\rho[\arg\omega(\sigma)]}{|\omega(\sigma)|}\cot\frac{\vartheta-\vartheta_0}{2}\,d\vartheta\right\}.$$

Here we should expect results similar to those established by S. E. Warschawski [141] in relation to Theodorsen's method.

function in the final form

$$\omega_1(\sigma) = \omega_0(\sigma)\, g_{n-1}(\sigma), \qquad g_{n-1}(\sigma) = G_{n-1}(\vartheta) + iG_{n-1}^{*}(\vartheta). \tag{44.15}$$

The numerical calculations were carried out for values of the semi-axes $a = \frac{2}{\sqrt{3}}$, $b = \frac{2}{\sqrt{5}}$; n was taken to be 12. The resulting expression for the first approximation is

$$\omega(\zeta) = \zeta e^{\alpha\zeta^2+\beta} \sum_{\nu=0}^{5} a_{2\nu}\zeta^{2\nu}, \tag{44.16}$$

where

$\alpha = 0.127706$, $a_0 = 0.976450$, $a_2 = -0.003604$, $a_4 = 0.022810$,
$\beta = 0.016136$, $a_6 = 0.003460$, $a_8 = 0.000703$, $a_{10} = 0.000134$.

The curve which is the image of the circumference by the first approximation (44.16) deviates from the given ellipse by not more than 0.006345, and the maximum deviation for the initial approximation is of the order of 0.048524. The first iteration thus brings the image of the zero approximation about eight times closer to the true contour. Note that the image found by the method of L. V. Kantorovich in the example under consideration departs from the accurate one by about 0.02 (L. V. Kantorovich and V. I. Krylov [57], p. 441).

CHAPTER FIVE

On Plane Problems of Unsymmetrical Elasticity

According to the conception of the brothers E. Cosserat and F. Cosserat [18], which takes account of the rotational interaction of material particles, it is necessary to introduce into consideration couple-stresses (moment per unit area), along with ordinary stresses (force per unit area), in the study of the state of stress in a deformable solid continuum.

The presence of couple-stresses implies the asymmetry of the (ordinary) stress tensor and leads to a consideration of strain gradients of high orders. In the case of the plane deformation of an isotropic body* there appears one additional elastic constant having the dimensions of length and relating couples to the curvatures produced by them.

The constitutive equations and the boundary conditions corresponding to a particular equilibrium state are of a more complex form in comparison with the classical case. The number of boundary conditions (geometric, static and mixed) necessary for determining the elastic equilibrium of a body is three. The number of possible combinations for the specification of these conditions is considerably greater than the number of the same combinations in classical elasticity.

The interest in the theory of couple-stresses has appreciably increased in recent years and its various aspects have become the subject of study by many authors.** Most of the presently available large number of relevant scientific publications are devoted to the general theoretical problems concerning the justification of the theory.

In the present chapter we give the basic relations of unsymmetrical elasticity theory and indicate the application of the techniques of complex function theory to the solution of boundary value problems in the simplest cases.

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* A situation which will be meant in all that follows.

** We shall mention the work of R. D. Mindlin [81, 82, 83], R. D. Mindlin and H. F. Tiersten [80], C. Truesdell and R. A. Toupin [134], R. A. Toupin [133]. Among the investigations of the Soviet authors may be mentioned the work of E. L. Aero and E. V. Kuvshinskii [1], and G. N. Savin [109]. A rather extensive bibliography on the subject may be found in a survey paper by G. N. Savin and Yu. N. Nemish [111].

45. Basic Relations in Plane Elasticity with Couple-stresses. Formulation of Boundary Value Problems

1°. To the usual equations of equilibrium of a body relating four components of stress is added one equation of equilibrium for the moments M_x and M_y. The system of equations of equilibrium is of the form here (in the absence of body forces)*

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}=0, \qquad \frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0, \qquad \frac{\partial M_x}{\partial x}+\frac{\partial M_y}{\partial y}+\tau_{xy}-\tau_{yx}=0. \tag{45.1}$$

In view of the asymmetry of the stress tensor the shearing stresses have symmetrical and antisymmetrical components, τ_s and τ_a:

$$\tau_s=\frac{1}{2}(\tau_{xy}+\tau_{yx}), \qquad \tau_a=\frac{1}{2}(\tau_{xy}-\tau_{yx}). \tag{45.2}$$

The symmetrical component τ_s, as in the classical case, produces the usual shearing strain ε_{xy}:

$$\tau_s=2\mu\varepsilon_{xy}, \qquad \varepsilon_{xy}=\frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right), \tag{45.3}$$

where μ is the shearing modulus, and the antisymmetrical component τ_a which is balanced by the couple-stresses rotates the element through an angle ω given by

$$\omega=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right). \tag{45.4}$$

The couple-stresses M_x, M_y will bend the element, giving rise to curvatures $\varkappa_x$, $\varkappa_y$ proportional, by hypothesis, to these stresses and defined by the formulas

$$\left.\begin{aligned} \varkappa_x&=\frac{\partial \omega}{\partial x}, & \varkappa_y&=\frac{\partial \omega}{\partial y}, \\ \varkappa_x&=\frac{1}{4B}M_x, & \varkappa_y&=\frac{1}{4B}M_y, \end{aligned}\right\} \tag{45.5}$$

where B is a constant called the flexural modulus.

The new material constant characterizing the degree to which the influence of the couple-stresses prevails is taken here to be a quantity l having the dimensions of length and defined by the equality

$$l^2=\frac{B}{\mu}. \tag{45.6}$$

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* The content of the present paragraph is taken from R. D. Mindlin's papers [81, 83].

The field of (usual and couple-) stresses is expressed in terms of two stress functions, U and V, by the formulas

$$\left.\begin{aligned} \sigma_x &= \frac{\partial^2 U}{\partial y^2} - \frac{\partial^2 V}{\partial x\,\partial y}, & \sigma_y &= \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 V}{\partial x\,\partial y}, \\ \tau_{xy} &= -\frac{\partial^2 U}{\partial x\,\partial y} - \frac{\partial^2 V}{\partial y^2}, & \tau_{yx} &= -\frac{\partial^2 U}{\partial x\,\partial y} + \frac{\partial^2 V}{\partial x^2}, \end{aligned}\right\} \tag{45.7}$$

$$M_x = \frac{\partial V}{\partial x}, \qquad M_y = \frac{\partial V}{\partial y}. \tag{45.8}$$

The stress functions U and V satisfy the system of equations

$$\frac{\partial}{\partial x}(V - l^2 \Delta V) = -2(1-\nu)\, l^2 \frac{\partial}{\partial y} \Delta U,$$

$$\frac{\partial}{\partial y}(V - l^2 \Delta V) = 2(1-\nu)\, l^2 \frac{\partial}{\partial x} \Delta U.$$

From these equations it will readily be noticed that the function U is a biharmonic one, and V satisfies the equation

$$\Delta V - l^2 \Delta^2 V = 0.$$

For the elastic displacements u, v and stresses (45.7), (45.8) there are the following relations of the general complex representation established by R. D. Mindlin [83]:*

$$2\mu(u + iv) = \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} - 8(1-\nu)\, l^2 \overline{\varphi''(z)} + 2i \frac{\partial H}{\partial \bar z}, \tag{45.9}$$

$$\left.\begin{aligned} &\sigma_x + \sigma_y = 2\left[\varphi'(z) + \overline{\varphi'(z)}\right], \\ &\sigma_y - \sigma_x + i(\tau_{xy} + \tau_{yx}) = 2\Big[\bar z \varphi''(z) + \psi'(z) + \\ &\qquad\qquad + 8(1-\nu)\, l^2 \varphi'''(z) + 2i \frac{\partial^2 H}{\partial z^2}\Big], \end{aligned}\right\} \tag{45.10}$$

$$\left.\begin{aligned} &\tau_{xy} - \tau_{yx} = -\lambda^2 H, \\ &M_x + iM_y = 8i(1-\nu)\, l^2 \overline{\varphi''(z)} + 2\frac{\partial H}{\partial \bar z}, \quad \varkappa = 3 - 4\nu. \end{aligned}\right\} \tag{45.11}$$

Here φ and ψ denote the Kolosov-Muskhelishvili potentials

$$2U = \bar z \varphi(z) + z\overline{\varphi(z)} + \chi(z) + \overline{\chi(z)}, \qquad \chi'(z) = \psi(z), \tag{45.12}$$

ν is Poisson's ratio, H is the solution of Helmholtz' equation

$$\Delta H - \lambda^2 H = 0, \qquad \lambda = \frac{1}{l} \tag{45.13}$$

and $\frac{\partial}{\partial \bar z}$ denotes the operation

$$\frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{45.14}$$

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* There are misprints in the cited work.

Finally, the function V admits the representation (op. cit.)

$$V = -4i(1-\nu)\, l^2 [\varphi'(z) - \overline{\varphi'(z)}] + H. \tag{45.15}$$

We write further the well-known relations between the elastic elements in systems of polar and Cartesian co-ordinates

$$\left.\begin{aligned} &\sigma_r + \sigma_\vartheta = \sigma_x + \sigma_y, \\ &\tau_{r\vartheta} - \tau_{\vartheta r} = \tau_{xy} - \tau_{yx}, \\ &\sigma_\vartheta - \sigma_r + i(\tau_{r\vartheta} + \tau_{\vartheta r}) = [\sigma_y - \sigma_x + i(\tau_{xy} + \tau_{yx})]\, e^{2i\vartheta}, \end{aligned}\right\} \tag{45.16}$$

$$M_r + iM_\vartheta = (M_x + iM_y)\, e^{-i\vartheta}, \tag{45.17}$$

$$v_r + iv_\vartheta = (u + iv)\, e^{-i\vartheta}, \qquad z = x + iy = re^{i\vartheta}. \tag{45.18}$$

On the basis of these formulas and formulas (45.9) to (45.11) we obtain the following complex representation formulas for the solutions of the equations of the theory of couple-stresses in polar co-ordinates:

$$\left.\begin{aligned} &\sigma_r + \sigma_\vartheta = 2[\varphi'(z) + \overline{\varphi'(z)}], \\ &\sigma_\vartheta - \sigma_r + i(\tau_{r\vartheta} + \tau_{\vartheta r}) = 2\Big[\bar{z}\varphi''(z) + \psi'(z) + \\ &\qquad\qquad + 8(1-\nu)\, l^2 \varphi'''(z) + 2i \frac{\partial^2 H}{\partial z^2}\Big] e^{2i\vartheta}, \\ &\tau_{r\vartheta} - \tau_{\vartheta r} = -\lambda^2 H, \end{aligned}\right\} \tag{45.19}$$

$$M_\vartheta + iM_r = \Big[8(1-\nu)\, l^2 \varphi''(z) + 2i \frac{\partial H}{\partial z}\Big] e^{i\vartheta}, \tag{45.20}$$

$$\begin{aligned} 2\mu(v_r + iv_\vartheta) = \Big[&\varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} - \\ &- 8(1-\nu)\, l^2 \overline{\varphi''(z)} + 2i \frac{\partial H}{\partial \bar{z}}\Big] e^{-i\vartheta}. \end{aligned} \tag{45.21}$$

2°. Let us calculate the components of the stress vector $X_n\, ds$, $Y_n\, ds$ acting on an elementary arc of some profile AB drawn in the xy plane. We have

$$\left.\begin{aligned} X_n &= \sigma_x \cos\widehat{nx} + \tau_{yx} \cos\widehat{ny}, \\ Y_n &= \tau_{xy} \cos\widehat{nx} + \sigma_y \cos\widehat{ny}, \end{aligned}\right\} \tag{45.22}$$

where n denotes the normal to the curve AB drawn to the right in relation to the positive sense on AB. Noting that on AB

$$\cos\widehat{nx} = \frac{dy}{ds}, \qquad \cos\widehat{ny} = -\frac{dx}{ds},$$

from formulas (45.22) and (45.7) we derive

$$X_n = \frac{d}{ds}\left(\frac{\partial U}{\partial y} - \frac{\partial V}{\partial x}\right), \qquad Y_n = -\frac{d}{ds}\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right).$$

By adding these equalities together, after multiplying the second by i, and integrating the resulting sum with respect to s, we obtain

$$2\frac{\partial}{\partial \bar{z}}(U-iV)=i\int_0^s (X_n+iY_n)\,ds+\text{constant}.$$

Substituting for U and V their expressions (45.12) and (45.15), we obtain the following equality valid along any profile in the physical region:

$$\varphi(z)+z\overline{\varphi'(z)}+\overline{\psi(z)}+8(1-\nu)\,l^2\overline{\varphi''(z)}-2i\frac{\partial H}{\partial \bar{z}}=$$

$$=i\int_0^s (X_n+iY_n)\,ds+\text{constant}. \qquad (45.23)$$

In the same way, proceeding from formulas (45.8) and using representation (45.15) for the couple-stress M_n acting on the same profile AB

$$M_n=M_x\cos\widehat{nx}+M_y\cos\widehat{ny}, \qquad (45.24)$$

we find

$$M_n=-\operatorname{Re}\left\{\frac{\overline{dt}}{ds}\left[8(1-\nu)\,l^2\overline{\varphi''(z)}-2i\frac{\partial H}{\partial \bar{z}}\right]\right\}. \qquad (45.25)$$

We present, finally, the complex representation formula for the angle of rotation ω given by equality (45.4). On the basis of formula (45.9) we have

$$\omega=\operatorname{Im}\left\{\frac{\partial}{\partial \bar{z}}(u+iv)\right\}=\frac{1}{2\mu}\operatorname{Im}\left\{(\varkappa+1)\varphi'(z)+\frac{i}{2l^2}H\right\}. \qquad (45.26)$$

Consider now the integral

$$I=\int_L (X_n u+Y_n v+M_n\omega)\,ds, \qquad (45.27)$$

where L denotes the contour of the physical region S which is assumed finite.

Replacing X_n, Y_n, M_n under the integral by their values from formulas (45.22), (45.24), we transform the integral according to Green's formula and make use of the equilibrium equations (45.1). We have

$$I=\iint_S\left[\sigma_x\frac{\partial u}{\partial x}+\sigma_y\frac{\partial v}{\partial y}+\tau_{xy}\frac{\partial v}{\partial x}+\tau_{yx}\frac{\partial u}{\partial y}+\right.$$

$$\left.+(\tau_{yx}-\tau_{xy})\,\omega+M_x\frac{\partial \omega}{\partial x}+M_y\frac{\partial \omega}{\partial y}\right]dx\,dy. \qquad (45.28)$$

Considering (45.4) and (45.3), we find

$$\tau_{xy}\frac{\partial v}{\partial x}+\tau_{yx}\frac{\partial u}{\partial y}+(\tau_{yx}-\tau_{xy})\,\omega=\tau_s\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)=4\mu\varepsilon_{xy}^2. \qquad (45.29)$$

If the sum of the first two terms under integral (45.28) is expressed according to Hooke's law in terms of the strains ε_x, ε_y, equality (45.29) is taken into account and formulas (45.5) are used for the last two terms under the integral, then the above integral is represented as

$$I=2\iint\limits_S W\,dx\,dy, \qquad (45.30)$$

where

$$2W=\lambda\,(\varepsilon_x+\varepsilon_y)^2+2\mu\,(\varepsilon_x+\varepsilon_y^2+2\varepsilon_{xy}^2)+$$
$$+4B\,(\varkappa_x^2+\varkappa_y^2). \qquad (45.31)$$

The expression for W, which is a non-singular positive quadratic form of the set of strain components (both usual and induced by couple-stresses), represents the potential energy or elastic strain energy per unit volume.

From this it follows immediately that all six components of stress, σ_x, σ_y, τ_{xy}, τ_{yx}, M_x, M_y, are zero throughout the region if either of the following two groups of conditions are fulfilled on its boundary:

$$X_n=Y_n=M_n=0, \qquad (45.32)$$

$$u=v=\omega=0. \qquad (45.33)$$

By conditions (45.32), the solution will be determinable apart from a rigid displacement of the body as a whole:

$$u=-\varepsilon y+\alpha, \qquad v=\varepsilon x+\beta,$$

where ε, α, β are constants.

By analogy with the classical case, the problems of determining the elastic equilibrium of a body subject to the non-homogeneous conditions (45.32) and (45.33) will be called, respectively, the first and second fundamental problems of couple-stress elasticity.

We have proved above the uniqueness of solutions of these problems for a finite region. The same reasoning also holds, with some limitation on the stress field at infinity, for the case of infinite regions.

As in the classical case, for the existence of the solution of the first fundamental problem in finite regions it is necessary that the conditions of the statics of a rigid body should be fulfilled. In our case these conditions are

$$\int\limits_L (X_n+iY_n)\,ds=0, \qquad \int\limits_L (xY_n-yX_n+M_n)\,ds=0. \qquad (45.34)$$

They follow directly from the equilibrium equations (45.1).

The foregoing complex representations of the fields of stress, displacement and strain enable us now to formulate the fundamental boundary value problems of couple-stress theory in terms of function theory.

First fundamental problem. *Find analytic functions* $\varphi(z)$, $\psi(z)$ *of the complex argument* z *in the region under consideration and a solution* H *of Eq.* (45.13) *regular in the same region from the boundary conditions*

$$\left.\begin{aligned} &\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} + 8(1-\nu)\, l^2\overline{\varphi''(t)} - \\ &\qquad\qquad - 2i\frac{\partial H}{\partial \bar{t}} = f_1(t) + if_2(t) + \text{constant}, \\ &\operatorname{Re}\left\{\frac{d\bar{t}}{ds}\left[8(1-\nu)\, l^2\overline{\varphi''(t)} - 2i\frac{\partial H}{\partial \bar{t}}\right]\right\} = f_0(t) \text{ on } L, \end{aligned}\right\} \tag{45.35}$$

where f_1, f_2, f_0 *are given real functions of* t *on the contour* L *which are defined in terms of the external forces and moments acting on the boundary of the body by the equalities*

$$f_1 + if_2 = i\int_0^s (X_n + iY_n)\, ds, \qquad f_0 = -M_n. \tag{45.36}$$

To the second fundamental problem correspond the boundary conditions

$$\left.\begin{aligned} &\varkappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)} - 8(1-\nu)\, l^2\overline{\varphi''(t)} + \\ &\qquad\qquad + 2i\frac{\partial H}{\partial \bar{t}} = 2\mu(g_1 + ig_2), \\ &\operatorname{Im}\left\{(\varkappa+1)\,\varphi'(t) + \frac{iH}{2l^2}\right\} = 2\mu g_0 \text{ on } L, \end{aligned}\right\} \tag{45.37}$$

where g_1, g_2 *are given displacements of the points of the contour* L, g_0 *denotes given values of the angle of rotation on* L:

$$u + iv = g_1 + ig_2, \qquad \omega = g_0 \text{ on } L. \tag{45.38}$$

Along with the conditions of the two fundamental problems above, one can, of course, specify other boundary conditions, for instance, assign u, v, M_n, etc. on the boundary of the region. Among these other problems we shall mention only the one which corresponds to contact between a body and a rigid profile of given shape in the absence of frictional forces (see Sec. 1). As a third boundary condition we assume here the presence of angles of rotation of the points of the boundary and write the conditions of the problem in the form

$$T = 0, \quad v_n = g(t), \qquad \omega = h(t) \text{ on } L, \tag{45.39}$$

where T is the tangential component of the stress vector and v_n is the normal displacement; $g(t)$ and $h(t)$ denote given functions on

L. This problem will be referred to, by convention, as the *third problem* of couple-stress elasticity.

Based on formulas (45.35) and (45.37), one can construct without much difficulty effective solutions of the fundamental problems for regions bounded by one circumference or two concentric circumferences. Some simple problems of this kind will be considered in the next sections.

46. Solution of the First Fundamental Problem for an Infinite Plane with a Circular Hole*

The centre of a hole of radius R is taken at the origin. Conditions (45.35) without a non-essential constant on the right-hand side assume in our case the form

$$\left.\begin{aligned} &\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}+m\overline{\varphi''(t)}-2i\frac{\partial H}{\partial \bar{t}}=f_1+if_2,\\ &\operatorname{Re}\left\{\frac{iR}{t}\left[m\overline{\varphi''(t)}-2i\frac{\partial H}{\partial \bar{t}}\right]\right\}=-f_0, \quad t=Re^{i\vartheta}, \end{aligned}\right\} \tag{46.1}$$

where

$$m=8(1-\nu)l^2. \tag{46.2}$$

Suppose that the field at infinity is uniform** and, in addition, the resultant of the forces applied to the contour of the hole is zero.

Assume in the region outside the contour of the hole

$$\varphi(z)=\Gamma z+\sum_{1}^{\infty} a_k z^{-k}, \qquad \psi(z)=\Gamma' z+\sum_{0}^{\infty} a'_k z^{-k} \quad (\Gamma=\bar{\Gamma}), \tag{46.3}$$

$$H(x, y)=\sum_{-\infty}^{\infty} h_k K_k(\lambda r)e^{ik\vartheta}, \qquad r=\sqrt{x^2+y^2}, \quad h_{-k}=\bar{h}_k, \tag{46.4}$$

where $K_n(\lambda r)$ is the Bessel function of the second kind of the imaginary argument (Macdonald's function).

Expressing the operation $\frac{\partial}{\partial \bar{z}}$ in a system of polar co-ordinates, applying it to the right-hand side of expression (46.4) and using the differentiation rule for the function K_n (see, for example, G. N. Watson [142]), we obtain

$$\frac{\partial H}{\partial \bar{z}}=-\frac{1}{2}\sum_{-\infty}^{\infty} \lambda h_{k-1}K_k(\lambda r)e^{ik\vartheta}. \tag{46.5}$$

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* The problem for a force-free hole in a uniform field was solved in the above-cited work of R. D. Mindlin [81]. The solutions of some other similar problems may be found in the work of Y. Weitsman [144] and R. J. Hartranft and G. C. Sih [33].

** The couple-stresses at infinity are universally taken to be zero: $M_x^{(\infty)}=M_y^{(\infty)}=0$.

The series for the derivatives of the function $\varphi(z)$ are obviously obtained by differentiating series (46.3) for $\varphi(z)$ the appropriate number of times with respect to z. Set up all derivatives of the required orders and assume, in addition, on the circumference

$$\left.\begin{aligned} f_1(t) + if_2(t) &= \sum_{-\infty}^{\infty} A_k e^{ik\vartheta}, \\ -f_0(t) &= \sum_{-\infty}^{\infty} \alpha_k e^{ik\vartheta}, \quad \alpha_{-k} = \bar\alpha_k. \end{aligned}\right\} \tag{46.6}$$

All preceding series are assumed to be uniformly convergent on the circumference, and on this assumption they are substituted in conditions (46.1). We have after some rearrangement

$$\sum_1^{\infty} a_k R^{-k} e^{-ik\vartheta} - \sum_3^{\infty} (k-2) R^{-k+2} [1 - m(k-1) R^{-2}] \bar a_{k-2} e^{ik\vartheta} + \\ + \sum_0^{\infty} \bar a'_k R^{-k} e^{ik\vartheta} + \lambda \sum_{-\infty}^{\infty} c_{k-1} K_k(\lambda r) e^{ik\vartheta} + 2\Gamma R e^{i\vartheta} + \\ + \bar\Gamma' R e^{-i\vartheta} = \sum_{-\infty}^{\infty} A_k e^{ik\vartheta}, \tag{46.7}$$

$$\operatorname{Re}\left\{ i\left[m \sum_2^{\infty} k(k-1) R^{-k-1} \bar a_{k-1} e^{ik\vartheta} \right] \right\} + \\ + \frac{\lambda i}{2} \sum_{-\infty}^{\infty} c_k [K_{k+1}(\lambda r) + K_{k-1}(\lambda r)] e^{ik\vartheta} = \sum_{-\infty}^{\infty} \alpha_k e^{ik\vartheta} \tag{46.8}$$

with

$$c_k = ih_k, \qquad c_{-k} = -\bar c_k.$$

Comparison of the coefficients of $e^{in\vartheta}$ in these equalities gives

$$\left.\begin{aligned} a'_0 - \lambda \bar c_1 K_0 &= A_0, \\ i\lambda K_1 c_0 &= \alpha_0, \\ i\lambda [K_0 + K_2] c_1 &= 2\alpha_1, \end{aligned}\right\} \tag{46.9}$$

$$(2-n) R^{-n+2} [1 - m(n-1) R^{-2}] \bar a_{n-2} + \bar a'_n R^{-n} + \lambda c_{n-1} K_n = A_n \quad (n \geqslant 1), \tag{46.10}$$

$$mn(n+1) R^{-n-2} \bar a_n + \lambda [K_{n+2} + K_n] c_{n+1} = -2i\alpha_{n+1} \quad (n \geqslant 1), \tag{46.11}$$

$$a_n R^{-n} - \lambda \bar c_{n+1} K_n = A_{-n} \qquad (n \geqslant 1). \tag{46.12}$$

With this way of writing the systems of equations, when $n = 1$, the first term on the left-hand side of relation (46.10) containing the factor $(n - 2)$ must be taken equal to zero and the right-hand sides, A_1 and A_{-1}, of equalities (46.10) and (46.12) must be replaced, respectively, by $A_1 - 2\Gamma R$ and $A_{-1} - \overline{\Gamma}' R$.

System (46.11), (46.12) determines the unknowns a_n, c_{n+1} $(n \geqslant 1)$, and system (46.10) all unknowns a'_n beginning with $n = 1$. The lacking unknowns a'_0, c_0, c_1 are determined from formula (46.9).

Solving systems (46.11) and (46.12) simultaneously, we find

$$\left.\begin{aligned} a_n &= \frac{2iK_n\overline{\alpha}_{n+1}R + RK_n^*A_{-n}}{\Delta_n}, \\ \overline{c}_{n+1} &= \frac{R^{-(n+1)}\,[2i\overline{\alpha}_{n+1}R^2 - mn\,(n+1)\,A_{-n}]}{\lambda\Delta_n} \end{aligned}\right\} \tag{46.13}$$

and

$$\left.\begin{aligned} \Delta_n &= R^{-(n+1)}\,[mn\,(n+1)\,K_n + R^2K_n^*], \\ K_n^* &= K_{n+2} + K_n \qquad (n \geqslant 1). \end{aligned}\right\} \tag{46.14}$$

Inserting these values in formula (46.10), we obtain

$$\begin{aligned} \overline{a}'_n = A_nR^n + (n-2)\,[R^2 - m\,(n-1)] \times \\ \times \frac{RK_{n-2}^*\overline{A}_{-n+2} - 2iK_{n-2}\alpha_{n-1}R}{\Delta_{n-2}} + \frac{K_nR}{\Delta_{n-2}}\,[2i\alpha_{n-1}R^2 + \\ + m\,(n-1)\,(n-2)\,\overline{A}_{-n+2}] \qquad (n \geqslant 3). \end{aligned} \tag{46.15}$$

We shall not write out the values of the coefficients a'_0, c_0, c_1. The unknowns a'_1, a'_2 are determined from formula (46.10) in terms of c_0 and c_1.

It will readily be seen that if the given functions f_1, f_2, f_0 on the boundary are reasonably smooth the foregoing series for the potentials φ, ψ, H and their derivatives are absolutely and uniformly convergent on the circumference $|\,z\,| = R$. For this it is sufficient to assume, for example, that the given functions have third-order derivatives with respect to the arc length s satisfying Dirichlet's conditions. The problem is solved.

In the particular case when no external forces are applied on the contour of the hole, i.e., when

$$A_n = \alpha_n = 0 \qquad (n = 0, \pm 1, \ldots),$$

there are altogether four coefficients, a_1, a'_1, a'_3, c_2, different from zero. For these coefficients we have

$$\left.\begin{aligned} a_1 &= -\frac{R^2\,(K_3 + K_1)\,\overline{\Gamma}'}{\Delta_1}, \qquad a'_1 = -2R^2\Gamma, \\ a'_3 &= \frac{R^2\,[(2m - R^2)\,(K_3 + K_1) - 2mK_3]\,\overline{\Gamma}'}{\Delta_1}, \\ c_2 &= \frac{2mR\Gamma'}{\lambda R^2\Delta_1}, \qquad \Delta_1 = 2mK_1R^{-2} + K_3 + K_1. \end{aligned}\right\} \tag{46.16}$$

The solution defined by coefficients (46.16) with given values of the constants Γ and Γ' was indicated in the work of R. D. Mindlin [81]. By passing in formulas (46.16) to the limit as $l \to 0$ and using the asymptotic formulas for Macdonald's functions (see, for example, G. N. Watson [142]), we obtain (the limiting values of the coefficients are denoted by the same letters)

$$a_1 = -R^2\overline{\Gamma}', \qquad a_1' = -2R^2\Gamma, \qquad a_3' = -R^4\overline{\Gamma}', \qquad c_2 = 0.$$

The above values determine the solution of the problem in the classical case. In particular, for a plate under uniaxial tension in the x direction ($\Gamma = p/4$, $\Gamma' = -p/2$) we have the well-known Kirsch solution (N. I. Muskhelishvili [93], Sec. 56a)

$$a_1 = \frac{R^2}{2}p, \qquad a_1' = -\frac{R^2}{2}p, \qquad a_3 = \frac{R^4}{2}p,$$

$$\varphi(z) = \frac{p}{4}\left(z + \frac{2R^2}{z}\right), \qquad \psi(z) = -\frac{p}{2}\left(z + \frac{R^2}{z} - \frac{R^4}{z^3}\right)$$

$$(H \equiv 0).$$

47. Examples

1°. **Normal pressure, shearing stress and moment applied uniformly over the edge of a circular hole.** As before, we denote by N and T the projections on the outward normal n and the tangent to the contour directed to the left (when viewed along n) of the vector representing the external stress applied along the edge of the hole. According to the condition of the problem, the quantities N, T and M_n are given and have constant values, and the stresses at infinity are zero. Noting that

$$N + iT = -(X_n + iY_n)\,e^{-i\vartheta}$$

we find

$$i\int_0^s (X_n + iY_n)\,ds = iR\int_0^\vartheta (X_n + iY_n)\,d\vartheta = -(N + iT)Re^{i\vartheta},$$

the arc length s being measured on the contour L in the sense leaving our infinite region on the right. Hence, according to formula (45.36)

$$f_1 + if_2 = (N + iT)\,Re^{i\vartheta}.$$

From this we obtain

$$A_1 = (N + iT)\,R, \qquad \text{all other } A_k = 0. \qquad (47.1)$$

Further, on the basis of formulas (45.36) and (46.6) we have

$$\alpha_0 = M_n, \qquad \text{all other } \alpha_k = 0. \qquad (47.2)$$

Solving the system of three equations (46.9) gives

$$a_0' = c_1 = 0 \qquad c_0 = \frac{M_n}{i\lambda K_1}, \tag{47.3}$$

whereupon from (46.13) we find

$$a_n = c_n = 0, \qquad n \geqslant 1. \tag{47.4}$$

Equality (46.10) for $n = 1$ gives $\overline{a_1'} + \lambda c_0 R K_1 = A_1 R$, from which, using (47.1) and (47.3), we determine

$$a_1' = (N - iT)\, R^2 - iM_n R. \tag{47.5}$$

As can easily be verified, all other a_k' are zero.

On the basis of formulas (47.3) and (47.5) the solution of our problem is of the form

$$\left.\begin{aligned} &\varphi(z) = 0, \qquad \psi(z) = \frac{(N - iT)\, R^2 - iM_n R}{z}, \\ &H(x, y) = -\frac{M_n}{\lambda}\,\frac{K_0(\lambda r)}{K_1(\lambda R)}. \end{aligned}\right\} \tag{47.6}$$

2°. **Concentrated force applied at a point of an infinite plane.** Adopting the common definition of a concentrated force (see, for example, N. I. Muskhelishvili [93], Sec. 56a, Par. 4), we assign on the edge of a circular hole the conditions

$$X_n = \frac{X}{2\pi R}, \qquad Y_n = \frac{Y}{2\pi R}, \qquad M_n = 0 \quad \text{on} \quad L, \tag{47.7}$$

where X, Y are given constants and R denotes, as before, the radius of the hole. Suppose that the stresses at infinitely remote parts of the plane are zero ($\Gamma = \Gamma' = 0$).

We first calculate the right-hand side of the first of equalities (45.35). On the basis of formulas (45.36) and (47.7), with the same rule for the measurement of the arc length s on L as in the preceding paragraph, we have in our case

$$f_1 + if_2 = -i \int^{s} (X_n + iY_n)\, ds = -\frac{X + iY}{2\pi} \ln \sigma + \\ + \text{constant} \qquad (\sigma = e^{i\vartheta}). \tag{47.8}$$

On the other hand, in view of the single-valuedness of displacements u and v we must have in the region outside the hole

$$\varphi(z) = -A \ln z + \varphi_0(z), \qquad \psi(z) = \varkappa \bar{A} \ln z + \psi_0(z), \tag{47.9}$$

$$A = \frac{X + iY}{2\pi(1 + \varkappa)} \qquad (\varkappa = 3 - 4\nu), \tag{47.10}$$

where φ_0, ψ_0 are holomorphic functions regular at infinity.

Insert now expressions (47.9) in the left sides of equalities (46.1) expressing the boundary conditions of the problem, transpose the known quantities containing the coefficient A to the right sides and take into account equality (47.8). After elementary computations we then arrive at the following problem for determining the functions φ_0, ψ_0 and H:

$$\varphi_0(t)+t\overline{\varphi_0'(t)}+\overline{\psi_0(t)}+m\overline{\varphi_0''(t)}-2i\frac{\partial H}{\partial \bar{t}}=\bar{A}e^{2i\vartheta}-\frac{m\bar{A}}{R^2}e^{-2i\vartheta},$$

$$\mathrm{Re}\left\{\frac{iR}{t}\left[m\overline{\varphi_0''(t)}-2i\frac{\partial H}{\partial \bar{t}}\right]\right\}=\frac{im}{2R^2}\left(Ae^{3i\vartheta}-\bar{A}e^{-3i\vartheta}\right). \qquad (47.11)$$

From this the Fourier coefficients in expansions (46.6) are found to be

$$\left.\begin{aligned} &A_2=\bar{A}, && A_{-2}=-\frac{m}{R^2}\bar{A},\\ &\alpha_3=\frac{imA}{2R^2}, && \alpha_{-3}=-\frac{im\bar{A}}{2R^2}; \end{aligned}\right\} \qquad (47.12)$$

all other A_k and α_k are zero.

Turning now to system (46.9) through (46.12) it is easily seen that all unknowns in it, apart from a_2, a_2', a_4' and c_3, are zero. For these unknowns we have

$$\left.\begin{aligned} &a_2'=\bar{A}_2R^2, \qquad a_2=\frac{2iK_2\bar{\alpha}_3+K_2^*A_{-2}}{R^{-1}\Delta_2},\\ &c_3=-\frac{2iR^2\alpha_3+6m\bar{A}_{-2}}{\lambda R^3\Delta_2},\\ &a_4'=2(R^2-3m)\,a_2-\lambda R^4K_4\bar{c}_3, \end{aligned}\right\} \qquad (47.13)$$

where Δ_n, K_n^* are given by formulas (46.14).

By using the well-known asymptotic formula

$$K_n(x)=\frac{2^{n-1}(n-1)!}{x^n}+o(x^{-n})$$

suitable for small x, we pass in expressions (47.13) to the limit as $R\rightarrow 0$. After simple computations we find (the limiting values of the coefficients are denoted by the same letters)

$$a_2'=a_4'=0, \qquad a_2=-\frac{8(1-\nu)\,l^2\bar{A}}{3-2\nu}, \qquad c_3=\frac{8(1-\nu)^2\,lA}{3-2\nu}. \qquad (47.14)$$

Thus, the potentials φ, ψ and H corresponding to a concentrated force (X, Y) applied at the point $z=0$ of an infinite plane are of the form $(z=x+iy=re^{i\vartheta})$

$$\left.\begin{aligned} &\varphi(z)=-\frac{X+iY}{2\pi(1+\varkappa)}\ln z+\frac{a_2}{z^2},\\ &\psi(z)=\frac{\varkappa(X-iY)}{2\pi(1+\varkappa)}\ln z,\\ &H(x,\ y)=K_3(\lambda r)\left[h_3e^{3i\vartheta}+\bar{h}_3e^{-3i\vartheta}\right], \end{aligned}\right\} \qquad (47.15)$$

where the coefficients a_2 and $h_3 = -ic_3$ are given by formulas (47.14).

According to formulas (47.14), as $l \to 0$, $a_2 = c_3 = 0$ and formulas (47.15) will give the potentials $\varphi(z)$, $\psi(z)$ corresponding to a concentrated force (at the point $z = 0$) in the classical case.

3°. **Concentrated couple.** In the example of Par. 1° of the present section we set $N = M_n = 0$. Then, according to formulas (47.6)

$$\varphi(z) = 0, \qquad \psi(z) = -\frac{iTR^2}{z}, \qquad H(x, y) = 0. \qquad (47.16)$$

In the case under consideration

$$X_n = T \sin \vartheta, \qquad Y_n = -T \cos \vartheta$$

and for the resultant moment of the forces applied to the contour of the hole about the origin we have

$$W = \int\limits_L (xY_n - yX_n)\, ds = -2\pi R^2 T.$$

By allowing the radius of the hole R to tend to zero so that the moment W remains constant, we find

$$\lim_{R \to 0} R^2 T = -\frac{W}{2\pi}$$

and formulas (47.16) give

$$\varphi(z) = 0, \qquad \psi(z) = \frac{iW}{2\pi} \frac{1}{z}, \qquad H(x, y) = 0 \qquad (|z| > R), \quad (47.17)$$

Functions (47.17) are the same as in the classical case (N. I. Muskhelishvili [93], Sec. 56a, Par. 5).

4°. **Concentrated moment.** In the formulas of the example of Par. 1° we now put $N = T = 0$. From formulas (47.6) we then obtain

$$\varphi(z) = 0, \qquad \psi(z) = -\frac{iM_n R}{z}, \qquad H(x, y) = -\frac{M_n}{\lambda K_1(\lambda R)} K_0(\lambda r). \qquad (47.18)$$

The resultant moment of the forces applied to the circumference L is

$$M = \int\limits_L M_{r\vartheta} ds = 2\pi R M_n. \qquad (47.19)$$

By reducing indefinitely the radius R and requiring that the magnitude of M remain constant in so doing, we find the equality

$$\lim_{R \to 0} R M_n = \frac{M}{2\pi}$$

which, with the other limiting equality

$$\lim_{x \to 0} x K_1(x) = 1,$$

for a concentrated moment (at the point $z = 0$) on the basis of formula (47.19) gives

$$\varphi(z) = 0, \qquad \psi(z) = -\frac{iM}{2\pi}\frac{1}{z}, \qquad H(x, y) = -MK_0(\lambda r) \qquad (|z| > R). \tag{47.20}$$

48. Solution of the Second Fundamental Problem for an Infinite Plane with a Circular Hole

The problem is solved similarly to the first fundamental problem treated in Sec. 46.

As in the case of the first problem, we proceed from representations (46.3), (46.4)* and assume, for $|z| = R$,

$$2\mu(g_1 + ig_2) = \sum_{-\infty}^{\infty} A_k e^{ik\vartheta}, \qquad 2\mu g_0 = \sum_{-\infty}^{\infty} \alpha_k e^{ik\vartheta}. \tag{48.1}$$

Substitution of the above series in the conditions of the problem (45.37) gives for the unknown coefficients a_n, a_n', c_n a system of equations of the form

$$\left.\begin{aligned} -\bar{a}_0' + \lambda\bar{c}_1 K_0 &= A_0, \\ \lambda^2 c_0 K_0 &= 2i\alpha_0 - (\varkappa + 1)(\Gamma - \bar{\Gamma}), \\ \lambda^2 c_1 K_1 &= 2i\alpha_1, \end{aligned}\right\} \tag{48.2}$$

$$\bar{a}_n' R^{-n} - (n-2) R^{-n}[R^2 - m(n-1)]\,\bar{a}_{n-2} + \lambda c_{n-1} K_n = -A_n, \tag{48.3}$$

$$\varkappa a_n R^{-n} + \lambda\bar{c}_{n+1} K_n = A_{-n}, \tag{48.4}$$

$$(\varkappa + 1)\, n a_n R^{-n-1} + \lambda^2 \bar{c}_{n+1} K_{n+1} = -2i\bar{\alpha}_{n+1} \qquad (n \geqslant 1). \tag{48.5}$$

For $n = 1$ the term on the left-hand side of expression (48.3) containing the factor $(n - 2)$ must be taken equal to zero, and A_1 and A_{-1} on the right-hand sides of formulas (48.3) and (48.4) must be replaced, respectively, by $A_1 - (x\Gamma - \bar{\Gamma})R$ and $A_{-1} + \bar{\Gamma}'R$.

The solution of system (48.4), (48.5) is represented as

$$a_n = \frac{\lambda K_{n+1} A_{-n} + 2iK_n \bar{\alpha}_{n+1}}{R^{-n}\Delta_n}, \tag{48.6}$$

$$c_{n+1} = \frac{2i\varkappa\alpha_{n+1} - (\varkappa + 1)\, n\bar{A}_{-n} R^{-1}}{\lambda\Delta_n} \qquad (n \geqslant 1), \tag{48.7}$$

where

$$\Delta_n = \varkappa\lambda K_{n+1} - (\varkappa + 1)\, nK_n R^{-1}.$$

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*In the second fundamental problem, as in the classical case, one must also assign the components X, Y of the resultant of the external forces applied to the contour of the hole. We assume that $X = Y = 0$.

The investigation of the determinant Δ_n presents no difficulty. For this it is sufficient to recall the well-known relation (G. N. Watson [142])

$$nK_n(x) = \frac{x}{2}[K_{n+1}(x) - K_{n-1}(x)].$$

On this basis we have

$$\Delta_n = R^{-1}[\varkappa\lambda RK_{n+1} - (\varkappa+1)nK_n] = \frac{\lambda}{2}[(\varkappa-1)K_{n+1} + \\ + (\varkappa+1)K_{n-1}] > 0 \qquad (n \geqslant 1).$$

The unknowns $\bar{a}'_n$ are determined from the derived values of a_n and c_n by relation (48.3) in the following form:

$$\bar{a}'_n R^{-n} = -A_n + (n-2)[1 - m(n-1)R^{-2}](\lambda K_{n-1}\bar{A}_{-n+2} - \\ - 2iK_{n-2}\alpha_{n-1})\Delta^{-1}_{n-2} - \\ - K_n[2i\varkappa\alpha_{n-1} - (\varkappa+1)(n-2)\bar{A}_{-n+2}R^{-1}]\Delta^{-1}_{n-2} \qquad (n \geqslant 3). \qquad (48.8)$$

Finally, the unknowns a'_0, c_0, c_1 are determined from relations (48.2), and a'_1 and a'_2 from (48.3). The problem is thus solved.

49. Examples

1°. Uniform normal and tangential displacements and uniform rotation of points of the hole contour. It is assumed that the stresses and rotation vanish at infinity, the resultant of the external forces applied to the boundary is zero, and constant values are given for v_r, v_ϑ and ω on the contour of a hole. Then

$$2\mu(g_1 + ig_2) = 2\mu(v_r + iv_\vartheta)e^{i\vartheta}, \quad 2\mu g_0 = 2\mu\varepsilon \quad (\omega = \varepsilon) \quad \text{on } L.$$

Consequently,

$$A_1 = 2\mu(v_r + iv_\vartheta), \qquad \alpha_0 = 2\mu\varepsilon \qquad (49.1)$$

and all other A_k and α_k are zero. Moreover, according to the condition of the problem, $\Gamma = \Gamma' = X = Y = 0$.

From relations (48.2) we determine the first three unknowns

$$c_1 = a'_0 = 0, \qquad c_0 = \frac{2i\alpha_0}{\lambda^2 K_0} \qquad (49.2)$$

and, further, from formulas (48.6) and (48.7) we find that $a_n = c_n = 0$ $(n \geqslant 1)$. Setting $n = 1$ in expression (48.3) and taking into account (49.2), we find a'_1 in the form

$$a'_1 = -R\left(\bar{A}_1 - \frac{2i\alpha_0 K_1}{\lambda K_0}\right);$$

the remaining a'_n are zero by formula (48.8). We have, finally,

$$\left.\begin{aligned} &\varphi(z)=0, \quad \psi(z)=\left[\frac{2i\alpha_0 K_1(\lambda R)}{\lambda K_0(\lambda R)}-\bar{A}_1\right]\frac{R}{z}, \\ &H(x, y)=\frac{2\alpha_0}{\lambda^2 K_0(\lambda R)}K_0(\lambda r) \qquad (|z|>R), \end{aligned}\right\} \tag{49.3}$$

where A_1 and α_0 are given by formulas (49.1).

We now calculate the resultant moment of the external forces applied to the contour L of the circular hole. The resultant moment M is obviously equal to

$$M=-\int_L (xY_n-yX_n+M_n)\,ds. \tag{49.4}$$

On the basis of equality (45.36)

$$\frac{df_1}{ds}=-Y_n, \quad \frac{df_2}{ds}=X_n,$$

f_1 and f_2 being single valued on L since the resultant of the external forces is zero. Hence,

$$\int_L (xY_n-yX_n)\,ds=\int_L (f_1-if_2)\,dt. \tag{49.5}$$

On the other hand, according to formula (45.25) and the first of equalities (45.35),

$$M_n=-\operatorname{Re}\left\{\left[m\varphi''(t)+2i\frac{\partial H}{\partial t}\right]\frac{dt}{ds}\right\},$$

$$f_1-if_2=\overline{\varphi(t)}+\bar{t}\varphi'(t)+\psi(t)+m\varphi''(t)+2i\frac{\partial H}{\partial t}+\text{constant on } L.$$

Inserting these expressions in formula (49.4) and taking into account (49.5), we obtain the following very simple formula for the moment M:

$$-M=\operatorname{Re}\int_L [\overline{\varphi(t)}+\bar{t}\varphi'(t)+\psi(t)]\,dt. \tag{49.6}$$

Formula (49.6) is always valid, of course, for finite singly connected regions. In our case this formula gives, on the basis of expressions (49.3),

$$M=4\pi\mu R\left[v_\vartheta+\frac{2\varepsilon K_1(\lambda R)}{\lambda K_0(\lambda R)}\right]. \tag{49.7}$$

2°. **Infinite plane with a rigid circular core.** Suppose that a rigid core of circular shape inserted into a hole in an infinite plate is bonded to the surrounding material along the contour of the hole, and the state of stress in the plate is produced by given forces and rotation at infinity characterized by constants Γ and Γ'. The resultant of the external forces applied to the boundary of the medium must be taken equal to zero.

We shall assume that the displacements and rotation of the points of the core contour L vanish,* and hence

$$g_1 = g_2 = g_0 = 0 \text{ on } L.$$

Only the coefficients containing Γ and Γ' then remain on the right-hand sides of system (48.2) to (48.5), and the non-vanishing unknowns in the system are a_1, a_1', a_3', c_0 and c_2. For these we have

$$\left.\begin{aligned} & a_1 = \frac{\lambda K_2 R^2 \overline{\Gamma}'}{\Delta_1}, \quad a_1' = (\varkappa\overline{\Gamma} - \Gamma) R^2 + (\varkappa + 1)(\overline{\Gamma} - \Gamma)\frac{K_1 R}{\lambda K_0} \\ & c_0 = -\frac{(\varkappa+1)(\Gamma - \overline{\Gamma})}{\lambda^2 K_0}, \qquad c_2 = -\frac{(\varkappa+1)\Gamma'}{\lambda\Delta_1}, \\ & a_3' = [(1 - 2mR^{-2})\lambda K_2 + (\varkappa+1) K_3 R^{-1}]\frac{R^4\overline{\Gamma}'}{\Delta_1}, \\ & \Delta_1 = \varkappa\lambda K_2 - (\varkappa+1) K_1 R^{-1}. \end{aligned}\right\} . \tag{49.8}$$

Passing to the limit as $l \to 0$ gives the values

$$a_1 = \frac{R^2\overline{\Gamma}'}{\varkappa}, \qquad a_1' = (\varkappa\overline{\Gamma} - \Gamma) R^2, \qquad a_3' = \frac{R^4\overline{\Gamma}'}{\varkappa}, \tag{49.9}$$
$$c_0 = c_2 = 0$$

which determine the solution of our problem in the classical case.

3°. **The case when a rigid core is acted on by a force applied to its centre.** Suppose now that in the conditions of the preceding paragraph $\Gamma = \Gamma' = 0$ and a given force (X, Y) is applied at the centre of the core $z = 0$. We may assume that the core does not move at all, i.e.

$$g_1 = g_2 = g_0 = 0 \qquad \text{on } L. \tag{49.10}$$

The resultant of the external forces exerted on the plate by the core has the components X and Y, and hence, as in Par. 2° of Sec. 47, the functions $\varphi(z)$ and $\psi(z)$ admit the representation

$$\varphi(z) = -A \ln z + \varphi_0(z), \qquad \psi(z) = x\overline{A} \ln z + \psi_0(z), \tag{49.11}$$

$$A = \frac{X + iY}{2\pi(1+\varkappa)}. \tag{49.12}$$

If expressions (49.11) are inserted in the boundary conditions (45.37) and condition (49.10) is taken into account, then, for the determination of the functions φ_0, ψ_0, H we obtain problem (45.37) with the right-hand sides

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* The solution of the problem in the case when the surrounding material imparts to the points of the contour L a tangential displacement and a rotation about the centre of the core presents no difficulty either.

$$2\mu\left(g_1^{(0)}+ig_2^{(0)}\right)=2\varkappa A\ln R-\overline{A}e^{2i\vartheta}+\frac{m\overline{A}}{R^2}e^{-2i\vartheta},$$

$$2\mu g_0^{(0)}=\frac{\varkappa+1}{2iR}\left[Ae^{-i\vartheta}-\overline{A}e^{i\vartheta}\right].$$

Consequently,

$$\left.\begin{aligned}&A_0=2\varkappa A\ln R,\quad A_2=-\overline{A},\quad A_{-2}=\frac{m\overline{A}}{R^2},\\&\alpha_1=-\frac{\varkappa+1}{2iR}\overline{A},\qquad \alpha_{-1}=\frac{\varkappa+1}{2iR}A.\end{aligned}\right\}\qquad(49.13)$$

By solving system (48.2) to (48.5) with free terms defined by expressions (49.13), we find

$$\left.\begin{aligned}&a_2=\frac{\lambda mK_3}{\Delta_2}\overline{A},\qquad -a_0'=\left[2\varkappa\ln R+\frac{(\varkappa+1)K_0}{RK_1}l\right]\overline{A},\\&a_2'=\left[R+\frac{(\varkappa+1)lK_2}{K_1}\right]RA,\qquad c_1=-\frac{(\varkappa+1)l^2}{RK_1}\overline{A},\\&a_4'=2\left[(R^2-3m)\lambda K_3+(\varkappa+1)RK_4\right]\frac{m\overline{A}}{\Delta_2},\\&c_3=-\frac{2(\varkappa+1)mA}{\lambda R^3\Delta_2},\qquad \Delta_2=\varkappa\lambda K_3-2(\varkappa+1)R^{-1}K_2.\end{aligned}\right\}\qquad(49.14)$$

The solution of the problem is

$$\varphi(z)=-A\ln z+\frac{a_2}{z^2},\qquad \psi(z)=\varkappa\overline{A}\ln z+a_0'+\frac{a_2'}{z^2}+\frac{a_4'}{z^4},$$

$$H(x,y)=2\mathrm{Re}\left[h_1K_1(\lambda r)e^{i\vartheta}+h_3K_3(\lambda r)e^{3i\vartheta}\right],$$

where the coefficients are given by equalities (49.14) and (49.12).*

50. Third Problem for a Plane with a Circular Hole**

As we saw above, the infinite systems of linear equations obtained in the solution of the first and second fundamental problems of couple-stress elasticity are solved in closed form, and this makes it possible to construct solutions of these problems in the form of power series. The same applies to the third problem of couple-stress elasticity which we shall now consider.

The boundary conditions of this problem (45.39) in the case under consideration are

$$\tau_{r\vartheta}=0,\qquad v_r=g(t),\qquad \omega=h(t)\qquad \text{on } L,\qquad(50.1)$$

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* It will be recalled that $c_k=ih_k$.

** The results of this section are presented in the author's paper [54].

where $g(t)$, $h(t)$ are given functions on the circumference L ($t = Re^{i\vartheta}$). On the basis of representations (45.19), (45.21) and (45.27) conditions (50.1) may be written as

$$\operatorname{Im}\left\{\left[\bar{t}\varphi''(t)+\psi'(t)+m\varphi'''(t)+2i\frac{\partial^2 H}{\partial t^2}\right]e^{2i\vartheta}-i\frac{\lambda^2}{2}H\right\}=0, \qquad (50.2)$$

$$\operatorname{Re}\left\{\left[\varkappa\overline{\varphi(t)}-\bar{t}\varphi'(t)-\psi(t)-m\varphi''(t)-2i\frac{\partial H}{\partial t}\right]e^{i\vartheta}\right\}=2\mu g(t), \qquad (50.3)$$

$$\operatorname{Im}\left\{(\varkappa+1)\varphi'(t)+\frac{i\lambda^2}{2}H\right\}=2\mu h(t) \qquad (t=Re^{i\vartheta}). \qquad (50.4)$$

In our case when the region is bounded by a circumference it appears possible to eliminate from conditions (50.2), (50.3) the boundary values of higher-order derivatives of φ, ψ and H. Indeed, on the basis of formula (45.21) and the last two relations of (45.19) we can write

$$\left[2\mu v_r+i\left(2\mu\frac{dv_r}{d\vartheta}-R\tau_{r\vartheta}\right)\right]e^{i\vartheta}=\varkappa\varphi(t)-t\overline{\varphi'(t)}-\overline{\psi(t)}-m\overline{\varphi''(t)}+$$
$$+2i\frac{\partial H}{\partial\bar{t}}-i(\varkappa+1)\bar{t}\operatorname{Im}\varphi'(t) \qquad (t=Re^{i\vartheta}).$$

The preceding equality enables us to write conditions (50.2) and (50.3) in the following equivalent form:

$$\varkappa\varphi(t)-t\overline{\varphi'(t)}-\overline{\psi(t)}-m\overline{\varphi''(t)}+2i\frac{\partial H}{\partial t}-i(\varkappa+1)t\operatorname{Im}\varphi'(t)=f(t), \qquad (50.5)$$

where

$$f(t)=2\mu[g(t)-tg'(t)]e^{i\vartheta} \qquad (t=Re^{i\vartheta}).$$

Equalities (50.5) and (50.4) constitute the boundary conditions of our problem.

To solve the problem, we make use of series* (46.3), (46.4) and assume, in addition, on the circumference

$$f(t)=\sum_{-\infty}^{\infty}A_k e^{ik\vartheta}, \qquad 2\mu h(t)=\sum_{-\infty}^{\infty}\alpha_k e^{ik\vartheta} \qquad (\alpha_{-k}=\bar{\alpha}_k). \qquad (50.6)$$

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* The components of the resultant of the external forces applied to the contour of the hole are taken to be zero.

For the determination of the coefficients a_k, a'_k, c_k $(c_k = ih_k)$ we then obtain the following system of linear equations:

$$\left.\begin{aligned} -\bar{a}'_0 + \lambda \bar{c}_1 K_0 &= A_0, \\ \lambda^2 c_0 K_0 &= 2i\alpha_0 - (\varkappa + 1)(\Gamma - \bar{\Gamma}), \\ \lambda^2 c_1 K_1 &= 2i\alpha_0, \end{aligned}\right\} \tag{50.7}$$

$$\left(\varkappa + \frac{\varkappa+1}{2} n\right) a_n R^{-n} + \lambda \bar{c}_{n+1} K_n = A_{-n} \qquad (n \geqslant 1), \tag{50.8}$$

$$(\varkappa + 1) n a_n R^{-(n+1)} + \lambda^2 \bar{c}_{n+1} K_{n+1} = -2i\bar{\alpha}_{n+1} \qquad (n \geqslant 1), \tag{50.9}$$

$$-\bar{a}'_n R^{-n} + (n-2)\left[\frac{\varkappa-1}{2} - m(n-1) R^{-2}\right] \bar{a}_{n-2} R^{-n+2} - \\ - \lambda c_{n-1} K_n = A_n \qquad (n \geqslant 1). \tag{50.10}$$

With this way of writing the systems of equations, when $n = 1$, the second term on the left-hand side of relation (50.10) must be taken equal to zero, and A_{-1} and A_1 on the right-hand sides of formulas (50.8) and (50.10) must be replaced, respectively, by $A_{-1} + \bar{\Gamma}' R$ and $A_1 + \frac{\varkappa-1}{2}(\Gamma + \bar{\Gamma}) R$.

By solving Eqs. (50.8) and (50.9) simultaneously, we find

$$a_n = \frac{\lambda K_{n+1} A_{-n} + 2i K_n \bar{\alpha}_{n+1}}{R^{-n} \Delta_n} \qquad (n \geqslant 1), \tag{50.11}$$

$$c_{n+1} = \frac{i\,[2\varkappa + (\varkappa+1) n]\, \alpha_{n+1} - (\varkappa + 1) n \bar{A}_{-n} R^{-1}}{\lambda \Delta_n} \qquad (n \geqslant 1), \tag{50.12}$$

$$\Delta_n = \left(\varkappa + \frac{\varkappa+1}{2} n\right) \lambda K_{n+1} - (\varkappa + 1) n K_n R^{-1}. \tag{50.13}$$

Here, too, the determinant Δ_n is different from zero. Indeed, on the basis of the well-known relation

$$n K_n(x) = \frac{x}{2}\,[K_{n+1}(x) - K_{n-1}(x)]$$

we have

$$\Delta_n > R^{-1}\,[\varkappa \lambda R K_{n+1} - (\varkappa + 1) n K_n] = \frac{\lambda}{2}\Big[(\varkappa - 1) K_{n+1} + \\ + (\varkappa + 1) K_{n-1}\Big] > 0 \qquad (n \geqslant 1). \tag{50.14}$$

The coefficients a'_n are now determined from the derived values of a_n and c_n by relation (50.10) as follows:

$$\bar{a}'_n R^{-n} = -A_n + (n-2)\left[\frac{\varkappa-1}{2} - m(n-1) R^{-2}\right] (\lambda K_{n-1} \bar{A}_{-n+2} - \\ - 2i K_{n-2} \alpha_{n-1})\, \Delta_{n-1}^{-1} + K_n \Big[(\varkappa + 1)(n-2) \bar{A}_{-n+2} R^{-1} - \\ - 2i\left\{\varkappa + \frac{\varkappa+1}{2}(n-2)\right\} \alpha_{n-1}\Big] \Delta_{n-2}^{-1} \qquad (n \geqslant 3). \tag{50.15}$$

The unknowns a'_0, c_0, c_1 are determined from system (50.7), and a'_1 and a'_2 directly from Eqs. (50.10) after c_0 and c_1 have been found.

Here again, on the basis of the asymptotic formulas for the functions K_n, when the subscript n is great, it can readily be verified that the foregoing series formed with the derived solution of system (50.7) to (50.10) are absolutely and uniformly convergent on the circumference if the given functions g and h on it are reasonably smooth. The problem is thus solved.

Examples. As an example, consider the case when the stresses in the medium are produced by a constant normal displacement of the points of the hole contour. The boundary conditions are

$$\tau_{r\vartheta} = 0, \qquad v_r = v_0, \qquad \omega = 0 \text{ on } L, \tag{50.16}$$

where v_0 is a given constant. Moreover, $\Gamma = \Gamma' = X = Y = 0$.

It is easy to see that in the case under consideration all coefficients A_k, α_k apart from A_1 are zero and

$$A_1 = 2\mu v_0.$$

Accordingly,

$$a'_1 = -2\mu v_0 R$$

and all other unknowns in system (50.7) to (50.10) are zero.

The solution is

$$\varphi(z) = 0, \qquad \psi(z) = -2\mu v_0 \frac{R}{z}, \qquad H(x, y) = 0 \qquad (|z| > R). \tag{50.17}$$

Consequently, in this case *no couple-stresses are induced in the medium.*

As another example, consider the case when

$$\tau_{r\vartheta} = v_r = \omega = 0 \text{ on } L \tag{50.18}$$

and the state of stress at infinity is determined by given constants Γ and Γ'.

In this case $A_k = \alpha_k = 0$ for all k ($k = 0, \pm 1, \pm 2, \ldots$), and there are altogether five unknowns left in system (50.7) to (50.10): a_1, a'_1, a'_3, c_0 and c_2. For these we have

$$\left.\begin{aligned}
&a_1 = \frac{\lambda K_2 R^2}{\Delta_1}\overline{\Gamma}', \quad a'_1 = \frac{\varkappa - 1}{2}(\Gamma + \overline{\Gamma}) R^2 + \frac{(\varkappa + 1) K_1}{\lambda K_0}(\overline{\Gamma} - \Gamma) R, \\
&a'_3 = \left[\left(\frac{\varkappa - 1}{2} - 2mR^{-2}\right)\lambda K_2 R^4 + (\varkappa + 1) K_3 R^3\right]\frac{\overline{\Gamma}'}{\Delta_1}, \\
&c_0 = \frac{(\varkappa + 1)(\overline{\Gamma} - \Gamma)}{\lambda^2 K_0}, \qquad c_2 = -\frac{(\varkappa + 1)\Gamma'}{\lambda \Delta_1}, \\
&\Delta_1 = \frac{3\varkappa + 1}{2}\lambda K_2 - (\varkappa + 1) K_1 R^{-1}.
\end{aligned}\right\} \tag{50.19}$$

These coefficients determine the solution of the problem in our case.

If in equalities (50.19) we pass to the limit as $l \to 0$ and make use of the asymptotic formulas for the functions $K_n\left(\frac{R}{l}\right)$ when their argument increases indefinitely, we obtain (the limiting values of the coefficients are denoted by the same letters)

$$a_1 = \frac{2R^2\overline{\Gamma}'}{3\varkappa+1}, \qquad a_1' = \frac{\varkappa-1}{2}(\Gamma+\overline{\Gamma})R^2, \qquad a_3' = \frac{\varkappa-1}{3\varkappa+1}R^4\overline{\Gamma}'. \tag{50.20}$$

$$c_0 = c_2 = 0.$$

These coefficients give the solution of the problem $\varphi(z)$, $\psi(z)$ in the classical case:

$$\left.\begin{aligned} &\varphi(z) = \Gamma z + \frac{a_1}{z}, \qquad \psi(z) = \Gamma' z + \frac{a_1'}{z} + \frac{a_3'}{z^3}, \\ &H(x, y) = 0 \qquad (|z| > R). \end{aligned}\right\} \tag{50.21}$$

51. On the Application of a Small Parameter

In the case of holes differing only slightly from a circle it is natural to try the small parameter method for the solution.

Let the contour of a hole in the plane $z = x + iy$ be given by the equation

$$\left.\begin{aligned} x &= a\left[\cos\vartheta + \varepsilon\sum_1^N(\alpha_n\cos n\vartheta + \beta_n\sin n\vartheta)\right], \\ y &= a\left[\sin\vartheta + \varepsilon\sum_1^N(-\alpha_n\sin n\vartheta + \beta_n\cos n\vartheta)\right], \end{aligned}\right\} \tag{51.1}$$

where a, α_n, β_n are constants, N is a natural number and ε is a small numerical parameter, $0 < \varepsilon < 1$, describing the deviation of the given hole from a circle.

Then the function

$$z = \omega(\zeta) = a[\zeta + \varepsilon f(\zeta)], \tag{51.2}$$

where

$$f(\zeta) = \sum_1^N(\alpha_n + i\beta_n)\zeta^{-n},$$

will conformally map the region outside the hole onto the region outside the unit circle in the plane of the variable $\zeta = \xi + i\eta$ provided

$$1 + \varepsilon f'(\zeta) \neq 0 \qquad (|\zeta| \geqslant 1). \tag{51.3}$$

We shall assume that condition (51.3) is fulfilled.

In the boundary conditions of the first fundamental problem (45.35) we write this time φ_1, ψ_1, H_1 for φ, ψ, H, respectively, and introduce the notation

$$\varphi(\zeta) = \varphi_1(z) = \varphi_1[\omega(\zeta)], \qquad \psi(\zeta) = \psi_1(z) = \psi_1[\omega(\zeta)],$$
$$\Phi(\zeta) = \varphi_1'(z) = \frac{\varphi'(\zeta)}{\omega'(\zeta)}, \qquad H(\zeta, \bar{\zeta}) = H_1(z, \bar{z}) = H_1[\omega(\zeta), \overline{\omega(\zeta)}], \tag{51.4}$$

where the function $\omega(\zeta)$ is given by equality (51.2).

After transformation of the variables the above-mentioned boundary conditions on the circumference of unit radius γ take the form

$$\left.\begin{aligned} &\overline{\omega'(\sigma)}\,\varphi(\sigma) + \omega(\sigma)\,\overline{\varphi'(\sigma)} + \overline{\omega'(\sigma)}\,\overline{\psi(\sigma)} + m\overline{\Phi'(\sigma)} - \\ &\qquad\qquad - 2i\frac{\partial H}{\partial \bar{\sigma}} = F(\sigma), \\ &\operatorname{Re}\left\{\frac{i}{\sigma}\left[m\,\overline{\Phi'(\sigma)} - 2i\,\frac{\partial H}{\partial \bar{\sigma}}\right]\right\} = h(\sigma) \qquad (\sigma = e^{i\vartheta}), \end{aligned}\right\} \tag{51.5}$$

where

$$\left.\begin{aligned} F(\sigma) &= \overline{\omega'(\sigma)}\,[f_1(\sigma) + if_2(\sigma)], \\ h(\sigma) &= -|\omega'(\sigma)|\,f_0(\sigma). \end{aligned}\right\} \tag{51.6}$$

The functions $\varphi(\zeta)$, $\psi(\zeta)$ are holomorphic in ζ outside γ (with certain reservations concerning their behaviour in the neighbourhood of the point at infinity) and $H(\zeta, \bar{\zeta})$ satisfies the equation

$$\Delta H - \lambda^2 |\omega'(\zeta)|^2 H = 0, \qquad \Delta = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} = 4\frac{\partial^2}{\partial \zeta\, \partial \bar{\zeta}}. \tag{51.7}$$

The potentials φ, ψ, H are sought in the form of power series in the parameter ε:

$$\left.\begin{aligned} &\varphi(\zeta) = \sum_0^\infty \varepsilon^n \varphi_n(\zeta), \qquad \psi(\zeta) = \sum_0^\infty \varepsilon^n \psi_n(\zeta), \\ &H(\xi, \eta) = \sum_0^\infty \varepsilon^n H_n(\xi, \eta), \end{aligned}\right\} \tag{51.8}$$

and the given functions are considered to be represented in the same form:

$$F(\sigma) = \sum_0^\infty \varepsilon^n F_n(\sigma), \qquad h(\sigma) = \sum_0^\infty \varepsilon^n h_n(\sigma). \tag{51.9}$$

Insert the expressions for the functions defined by formulas (51.8) and (51.9) in conditions (51.5) and compare the coefficients of like powers of ε. Then the following sequence of boundary value problems for a circular hole will result:

$$\left.\begin{aligned}&\varphi_0(\sigma)+\sigma\overline{\varphi_0'(\sigma)}+\overline{\psi_0(\sigma)}+\frac{m}{a}\overline{\Phi_0'(\sigma)}-\frac{2i}{a}\frac{\partial H_0}{\partial\bar\sigma}=\frac{1}{a}F_0(\sigma),\\&\operatorname{Re}\left\{\frac{i}{\sigma}\left[m\overline{\Phi_0'(\sigma)}-2i\frac{\partial H_0}{\partial\bar\sigma}\right]\right\}=h_0(\sigma),\end{aligned}\right\}\qquad(51.10)$$

$$\left.\begin{aligned}&\varphi_n(\sigma)+\sigma\overline{\varphi_n'(\sigma)}+\overline{\psi_n(\sigma)}+\frac{m}{a}\overline{\Phi_n'(\sigma)}-\frac{2i}{a}\frac{\partial H_n}{\partial\bar\sigma}+\\&+\varphi_{n-1}(\sigma)\overline{f'(\sigma)}+\overline{\varphi_{n-1}'(\sigma)f(\sigma)}+\overline{\psi_{n-1}(\sigma)f'(\sigma)}=\frac{1}{a}F_n(\sigma),\\&\operatorname{Re}\left\{\frac{i}{\sigma}\left[m\overline{\Phi_n'(\sigma)}-2i\frac{\partial H_n}{\partial\bar\sigma}\right]\right\}=h_n(\sigma)\qquad(n=1,\ 2,\ \ldots),\end{aligned}\right\}\qquad(51.11)$$

where

$$\Phi_n'(\zeta)=\frac{d}{d\zeta}\left\{\frac{\varphi_n'(\zeta)}{\omega'(\zeta)}\right\}.$$

The equation for the determination of H_n $(n=0,\ 1,\ \ldots)$ is obtained from (51.7), on inserting representation (51.8) for H and expression (51.2). We have

$$\left.\begin{aligned}&\Delta H_0-a^2\lambda^2H_0=0,\\&\Delta H_1-a^2\lambda^2H_1=a^2\lambda^2[f'(\zeta)+\overline{f'(\zeta)}]\,H_0,\\&\Delta H_n-a^2\lambda^2H_n=\\&\quad=a^2\lambda^2\{[f'(\zeta)+\overline{f'(\zeta)}]\,H_{n-1}+|f'(\zeta)|^2H_{n-2}\}\qquad(n\geqslant2).\end{aligned}\right\}\qquad(51.12)$$

After finding particular solutions of Eqs. (51.12) in appropriate form, problems (51.11) can be solved by the power series method (see. Sec. 46).

Example. Extension of a plate with an elliptical hole.* Consider the problem of stress distribution near an elliptical hole in the field of simple tension. Suppose that the contour L of the hole is free from external forces and moments, and the following state of stress is realized at infinity:

$$\sigma_y^{(\infty)}=p,\qquad\sigma_x^{(\infty)}=\tau_{xy}^{(\infty)}=\tau_{yx}^{(\infty)}=0,\qquad(51.13)$$

where p is a given constant.

The mapping function is here of the form

$$z=\omega(\zeta)=R\left(\zeta+\frac{\varepsilon}{\zeta}\right),\qquad\zeta=\rho e^{i\vartheta}\qquad(R>0,\ 0<\varepsilon<1).\qquad(51.14)$$

To the circumference $|\zeta|=\rho=1$ corresponds an ellipse L, centred at the origin, whose semi-axes are equal to

$$a=R\,(1+\varepsilon),\qquad b=R\,(1-\varepsilon).$$

---

* The example has been worked out by V. S. Zhgenti. See A. I. Kalandiya and V. S. Zhgenti [52].

The functions of the zero approximation $\varphi_0(\zeta)$, $\psi_0(\zeta)$, H_0 have already been constructed in Sec. 46. These are as follows:

$$\left.\begin{aligned}&\varphi_0(\zeta)=\Gamma\zeta+\frac{a_1^{(0)}}{\zeta}, \quad \psi_0(\zeta)=\Gamma'\zeta+\frac{b_1^{(0)}}{\zeta}+\frac{b_3^{(0)}}{\zeta^3},\\ &H_0=[h_2^{(0)}e^{2i\vartheta}+\bar{h}_2^{(0)}e^{-2i\vartheta}]\,K_2(\lambda R\rho),\end{aligned}\right\}\tag{51.15}$$

where

$$a_1^{(0)}=-\frac{\Gamma'}{R_0}, \quad b_1^{(0)}=-2\Gamma, \quad b_3^{(0)}=-\frac{R_2}{R_0}\Gamma', \quad h_2^{(0)}=-i\frac{R_1\Gamma'}{\lambda^3R^2R_0}\tag{51.16}$$

with

$$R_0=1+\frac{16(1-\nu)K_1}{\lambda^2R^2\,]K_1+K_3]}, \qquad R_1=\frac{16(1-\nu)}{K_1+K_3}, \qquad R_2=1-\frac{16(1-\nu)K_1}{\lambda^2R^2\,[K_1+K_3]}.$$

If the functions defined by expressions (51.15) are inserted, with (51.16), in Eqs. (51.12), the equation for H_1 is obtained in the form

$$\Delta H_1-\lambda^2R^2H_1=-\frac{2R_1\Gamma'}{\lambda R_0\rho^2}K_2(\lambda R\rho)\sin 4\vartheta.\tag{51.17}$$

On the basis of the identity

$$\frac{d^2K_n(\lambda R\rho)}{d\rho^2}+\frac{1}{\rho}\frac{dK_n(\lambda R\rho)}{d\rho}-\left(\lambda^2R^2+\frac{n^2}{\rho^2}\right)K_n(\lambda R\rho)=0\tag{51.18}$$

it is easily verified that the function

$$H_1^{(1)}=\frac{R_1\Gamma'}{6\lambda R_0}K_2(\lambda R\rho)\sin 4\vartheta\tag{51.19}$$

satisfies Eq. (51.17). Consequently, any solution of Eq. (51.17) may be represented as

$$H_1=H_1^{(0)}+H_1^{(1)},\tag{51.20}$$

where $H_1^{(0)}$ is the solution of the corresponding homogeneous equation and $H_1^{(1)}$ is defined by equality (51.19).

The functions $\varphi_1(\zeta)$, $\psi_1(\zeta)$, $H_1^{(0)}$ are now sought in the form

$$\left.\begin{aligned}&\varphi_1(\zeta)=\sum_{n=1}^{2}\frac{a_{2n-1}^{(1)}}{\zeta^{2n-1}}, \qquad \psi_1(\zeta)=\sum_{n=1}^{3}\frac{b_{2n-1}^{(1)}}{\zeta^{2n-1}},\\ &H_1^{(0)}=\sum_{n=1}^{2}[h_{2n}^{(1)}e^{2in\vartheta}+\bar{h}_{2n}^{(1)}e^{-2in\vartheta}]\,K_{2n}(\lambda R\rho).\end{aligned}\right\}\tag{51.21}$$

Inserting (51.21), (51.20) and (51.19) in (51.11) for $n = 1$ and equating to zero the coefficients of $e^{in\vartheta}$, we obtain

$$\left.\begin{aligned}
&a_1^{(1)} + i\lambda K_1 \overline{h}_2^{(1)} = -\Gamma,\\
&a_3^{(1)} + i\lambda K_3 \overline{h}_4^{(1)} = \frac{R_1\Gamma'}{12R_0}\left[K_1 - \frac{2}{\lambda R} K_2\right],\\
&\overline{b}_1^{(1)} = \Gamma' + 2a_1^{(0)},\\
&-\left[1 - \frac{16(1-\nu)}{\lambda^2R^2}\right]\overline{a}_1^{(1)} + \overline{b}_3^{(1)} + i\lambda K_3 h_2^{(1)} =\\
&\qquad\qquad = \Gamma + b_1^{(0)} + \frac{16(1-\nu)}{\lambda^2R^2}\Gamma,\\
&-3\left[1 - \frac{32(1-\nu)}{\lambda^2R^2}\right]\overline{a}_3^{(1)} + \overline{b}_5^{(1)} + i\lambda K_5 h_4^{(1)} =\\
&\qquad\qquad = b_3^{(0)} - \frac{32(1-\nu)}{\lambda^2R^2} a_1^{(0)} - \frac{R_1\Gamma'}{12R_0}\left[K_3 + \frac{2}{\lambda R} K_2\right],\\
&i\frac{16(1-\nu)}{\lambda^2R^2}\overline{a}_1^{(1)} - \lambda[K_1 + K_3]\,h_2^{(1)} = i\frac{16(1-\nu)}{\lambda^2R^2}\Gamma,\\
&\qquad i\frac{96(1-\nu)}{\lambda^2R^2}\overline{a}_3^{(1)} - \lambda[K_3 + K_5]\,h_4^{(1)} =\\
&\qquad\qquad = -i\frac{32(1-\nu)}{\lambda^2R^2} a_1^{(0)} - i\frac{R_1\Gamma'}{12R_0}[K_1 + K_3].
\end{aligned}\right\} \quad (51.22)$$

Solving system (51.22) gives

$$a_1^{(1)} = -\frac{R_2}{R_0}\Gamma, \qquad a_3^{(1)} = \frac{4(1-\nu)R_4\Gamma'}{3R_0R_3},$$

$$b_1^{(1)} = \left(1 - \frac{2}{R_0}\right)\Gamma', \qquad b_3^{(1)} = -\frac{2R_6}{R_0}\Gamma,$$

$$b_5^{(1)} = \frac{R_7}{R_0}\Gamma', \qquad h_2^{(1)} = -i\frac{2R_1\Gamma}{\lambda^3R^2R_0},$$

$$h_4^{(1)} = i\frac{4(1-\nu)R_5}{3\lambda R_0}\Gamma',$$

where

$$R_3 = 1 + \frac{96(1-\nu)K_3}{\lambda^2R^2[K_3+K_5]}, \qquad R_4 = \frac{K_1 - \frac{2}{\lambda R}K_2}{K_1+K_3} + \frac{\left(\frac{24}{\lambda^2R^2} - 1\right)K_3}{K_3+K_5},$$

$$R_5 = \frac{1 + \frac{24}{\lambda^2R^2}\left[4(1-\nu)\frac{R_4}{R_3} - 1\right]}{K_3+K_5} \qquad R_6 = 1 - \frac{16(1-\nu)}{\lambda^2R^2} + \frac{R_1K_3}{\lambda^2R^2},$$

$$R_7 = 4(1-\nu)\left[1 - \frac{32(1-\nu)}{\lambda^2R^2}\right]\frac{R_4}{R_3} + \frac{4}{3}(1-\nu)R_5K_5 - R_2 +$$
$$+ \frac{32(1-\nu)}{\lambda^2R^2} - \frac{R_1}{12}\left[K_3 + \frac{2}{\lambda R}K_2\right].$$

Inserting the zero and first approximations constructed above in Eq. (51.12) for $n = 2$ and using the formula

$$K_n(\lambda R\rho) = \frac{\lambda R\rho}{2n}\left[K_{n+1}(\lambda R\rho) - K_{n-1}(\lambda R\rho)\right], \tag{51.23}$$

we find

$$\begin{aligned}\Delta H_2 - \lambda^2 R^2 H_2 = &- \frac{4R_1\Gamma}{\lambda R_0\rho^2} K_2(\lambda R\rho)\sin 4\vartheta + \\ &+ 8(1-\nu)\frac{\lambda R^2 R_5}{3R_0\rho^2}\Gamma' K_4(\lambda R\rho)(\sin 2\vartheta + \sin 6\vartheta) - \\ &- \frac{\lambda R^2 R_1\Gamma'}{6R_0\rho^2} K_2(\lambda R\rho)(\sin 2\vartheta + \sin 6\vartheta) + \\ &+ \frac{\lambda R^2 R_1 \Gamma'}{4R_0\rho^2}\left[\frac{1}{3}K_4(\lambda R\rho) - \frac{4}{3}K_2(\lambda R\rho) + K_0(\lambda R\rho)\right]\sin 2\vartheta. \end{aligned} \tag{51.24}$$

Consider the equation

$$\Delta\omega - \lambda^2 R^2 \omega = \frac{1}{\rho^2} K_{2p}(\lambda R\rho)\sin 2q\vartheta \tag{51.25}$$

$$(p = 0,\ 1,\ 2,\ \ldots;\qquad q = 1,\ 2,\ \ldots).$$

A particular solution of this equation is sought in the form

$$\omega = V_{p,\,q}(\rho)\, K_{2q}(\lambda R\rho)\sin 2q\vartheta, \tag{51.26}$$

where $V_{p,\,q}$ is the unknown function. We have for it the following second-order differential equation:

$$\frac{d^2 V_{p,\,q}}{d\rho^2} + \frac{d}{d\rho}\ln\left[\rho K_{2q}^2(\lambda R\rho)\right]\frac{dV_{p,\,q}}{d\rho} = \frac{K_{2p}(\lambda R\rho)}{\rho^2 K_{2q}(\lambda R\rho)}. \tag{51.27}$$

By integration of this equation the particular solution may be obtained as

$$V_{p,\,q}(\rho) = \int_1^{\rho} \frac{F_{2p,\,2q-1}(\lambda Rt) - F_{2p,\,2q+1}(\lambda Rt)}{16qtK_{2q}^2(\lambda Rt)}\,dt, \tag{51.28}$$

where

$$F_{p,\,q}(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2x\cosh t\cosh\tau}\,\frac{\cosh(p+q)t\cosh(p-q)\tau}{\cosh t\cosh\tau}\,dt\,d\tau. \tag{51.29}$$

In constructing solution (51.28), use has been made of relation (51.23) and the well-known Nicholson integral representation (see G. N. Watson [142], p. 484)

$$K_p(x)K_q(x) = \\ = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2x\cosh t\cosh\tau}\cosh(p+q)t\cosh(p-q)\tau\,dt\,d\tau. \tag{51.30}$$

By using the obvious equalities

$$
\begin{aligned}
\cosh 3x &= \cosh x \cdot (2\cosh 2x - 1),\\
\cosh 5x &= \cosh x \cdot (2\cosh 4x - 2\cosh 2x + 1),\\
\cosh 7x &= \cosh x \cdot (2\cosh 6x - 2\cosh 4x + 2\cosh 2x - 1),\\
\cosh 9x &= \cosh x \cdot (2\cosh 8x - 2\cosh 6x + 2\cosh 4x - 2\cosh 2x + 1)
\end{aligned}
$$

and representation (51.30), it is easy to express the necessary number of first $F_{p,\,q}$ functions in terms of the Bessel functions as follows:

$$
\begin{aligned}
&F_{0,1} = 2K_0^2, \quad F_{0,3} = 8(K_0K_2 - K_1^2) + 2K_0^2,\\
&F_{2,1} = 2(2K_1^2 - K_0^2), \quad F_{2,3} = 2(2K_2^2 - 2K_1^2 + K_0^2),\\
&F_{4,1} = 2(4K_1K_3 + 4K_1^2 - 4K_0K_2 - 2K_2^2 - K_0^2),\\
&F_{4,3} = 2(2K_3 - 2K_2^2 + 2K_1^2 - K_0^2),\\
&F_{2,5} = 2(4K_2K_4 - 4K_1K_3 + 4K_0K_2 - 4K_1^2 - 2K_3^2 + 2K_2^2 + K_0^2),\\
&F_{2,7} = 8K_2K_6 - 8K_1K_5 + 8K_0K_4 + 8K_2K_4 + 8K_0K_2 -\\
&\qquad - 8K_1^2 + 4K_4^2 - 4K_3^2 + 2K_0^2 - 16K_3K_1 + 8K_2^2 - 8K_5K_3,\\
&F_{4,5} = 4K_4^2 - 4K_3^2 + 4K_2^2 - 4K_1^2 + 2K_0^2,\\
&F_{4,7} = 8K_4K_6 - 8K_3K_5 + 8K_2K_4 - 8K_1K_3 + 8K_0K_2 -\\
&\qquad - 8K_1^2 - 4K_5^2 + 4K_4^2 - 4K_3^2 + 4K_2^2 + 2K_0^2.
\end{aligned}
$$

It may be shown that

$$\lim_{x\to\infty} \frac{F_{2p,\,2q-1}(x) - F_{2p,\,2q+1}(x)}{K_{2q}^2(x)} = 0, \qquad (51.31)$$

$$\lim_{x\to\infty} \frac{x\,[F_{2p,\,2q-1}(x) - F_{2p,\,2q+1}(x)]}{K_{2q}^2(x)} = -8q. \qquad (51.32)$$

Based on equalities (51.31), (51.32), (51.28), (51.27) and also on the equalities

$$\lim_{x\to\infty} \frac{K_p(x)}{K_q(x)} = 1, \qquad \lim_{x\to\infty} K_p(x) = 0,$$

it is easily proved that the function ω defined by expression (51.26) tends to zero as $\rho \to \infty$ with its derivatives of the first two orders.

From the above discussion it follows that the required solution of Eq. (51.24) is representable in the form

$$H_2 = H_2^{(0)} + H_2^{(1)}, \qquad (51.33)$$

where $H_2^{(0)}$ satisfies the corresponding homogeneous equation, and

$$\begin{aligned}H_2^{(1)} = {} & \frac{\lambda R^2\Gamma'}{R_0}\left\{\frac{8}{3}(1-\nu)R_5V_{2,1}(\rho)-\frac{1}{6}R_1V_{1,1}(\rho)+\right.\\ & \left.+R_1\left[\frac{1}{12}V_{2,1}(\rho)-\frac{1}{3}V_{1,1}(\rho)+\frac{1}{4}V_{0,1}(\rho)\right]\right\}K_2(\lambda R\rho)\sin 2\vartheta-\\ & -\frac{4R_1\Gamma}{\lambda R_0}V_{1,2}(\rho)K_4(\lambda R\rho)\sin 4\vartheta+\frac{\lambda R^2\Gamma'}{R_0}\times\\ & \times\left[\frac{8}{3}(1-\nu)R_5V_{2,3}(\rho)-\frac{R_1}{6}V_{1,3}(\rho)\right]K_6(\lambda R\rho)\sin 6\vartheta. \qquad (51.34)\end{aligned}$$

The functions of the second approximation, $\varphi_2(\zeta)$, $\psi_2(\zeta)$, $H_2^{(0)}$, are sought in the form

$$\left.\begin{aligned}&\varphi_2(\zeta)=\sum_{n=1}^{3}\frac{a_{2n-1}^{(2)}}{\zeta^{2n-1}}, \qquad \psi_2(\zeta)=\sum_{n=1}^{4}\frac{b_{2n-1}^{(2)}}{\zeta^{2n-1}},\\ &H_2^{(0)}=\sum_{n=1}^{3}\left[h_{2n}^{(2)}e^{2in\vartheta}+\bar{h}_{2n}^{(2)}e^{-2in\vartheta}\right]K_{2n}(\lambda R\rho).\end{aligned}\right\} \qquad (51.35)$$

Substituting expressions (51.35), (51.33), (51.21) in relation (51.11) for $n=2$ and solving the corresponding system of algebraic equations, we obtain

$$a_1^{(2)}=\frac{R_{11}}{R_0^2}\Gamma', \qquad a_3^{(2)}=\frac{R_{13}\Gamma}{R_0R_3}, \qquad a_5^{(2)}=\frac{R_{16}\Gamma'}{R_0R_{15}},$$

$$b_1^{(2)}=-\frac{2R_2\Gamma}{R_0}, \qquad b_3^{(2)}=\frac{R_{18}\Gamma'}{R_0}, \qquad b_5^{(2)}=\frac{R_{19}\Gamma}{R_0},$$

$$b_7^{(2)}=\frac{R_{20}\Gamma'}{R_0}, \qquad h_2^{(2)}=i\frac{R_{12}\Gamma'}{\lambda R_0}, \qquad h_4^{(2)}=i\frac{R_{14}\Gamma}{\lambda R_0}, \qquad h_6^{(2)}=i\frac{R_{17}\Gamma'}{\lambda R_0},$$

where

$$R_8=\frac{(1-\nu)[F_{2,5}-F_{2,3}]}{\lambda R[K_1+K_3]K_4},$$

$$\begin{aligned}R_9= {} & \frac{\lambda R}{12}(1-\nu)\left\{1+\frac{24}{\lambda^2R^2}\left[4(1-\nu)\frac{R_4}{R_3}-1\right]\right\}\frac{F_{4,1}-F_{4,3}}{[K_3+K_5]K_2}+\\ & +\frac{\lambda R}{4}(1-\nu)\frac{\frac{1}{6}[F_{4,1}-F_{4,3}]-[F_{2,1}-F_{2,3}]+\frac{1}{2}[F_{0,1}-F_{0,3}]}{[K_1+K_3]K_2},\end{aligned}$$

$$\begin{aligned}R_{10}= {} & \frac{\lambda R}{36}(1-\nu)\left\{1+\frac{24}{\lambda^2R^2}\left[4(1-\nu)\frac{R_4}{R_3}-1\right]\right\}\frac{F_{4,5}-F_{4,7}}{[K_3+K_5]K_6}-\\ & -\frac{\lambda R}{36}(1-\nu)\frac{F_{2,5}-F_{2,7}}{[K_1+K_3]K_6},\end{aligned}$$

$$R_{11}=\frac{[K_1-K_3]R_9}{K_1+K_3}+\frac{4}{3}(1-\nu)\frac{R_4}{R_3}, \qquad R_{12}=\frac{2\left[\frac{8(1-\nu)R_{11}}{\lambda^2R^2R_0}-R_9\right]}{K_1+K_3},$$

$$R_{13}=\frac{64(1-\nu)K_3}{\lambda^2R^2[K_3+K_5]}+\frac{[K_3-K_5]R_8}{K_3+K_5},$$

$$R_{14}=\frac{2\left[\frac{16(1-\nu)}{\lambda^2R^2}\left(\frac{3R_{13}}{R_3}-2\right)-R_8\right]}{K_3+K_5},\qquad R_{15}=1+\frac{240(1-\nu)K_5}{\lambda^2R^2[K_5+K_7]},$$

$$R_{16}=\frac{[K_5-K_7]R_{10}}{K_5+K_7}+\frac{48(1-\nu)K_5\left[1-4(1-\nu)\frac{R_4}{R_3}\right]}{\lambda^2R^2[K_5+K_7]},$$

$$R_{17}=\frac{2}{K_5+K_7}\left\{\frac{24(1-\nu)}{\lambda^2R^2}\left[\frac{5R_{16}}{R_{15}}-1+4(1-\nu)\frac{R_4}{R_3}\right]-R_{10}\right\},$$

$$R_{18}=\left[1-\frac{16(1-\nu)}{\lambda^2R^2}\right]\frac{R_{11}}{R_0}+R_{12}K_3+4(1-\nu)\frac{R_4}{R_3}+R_0-2+R_9,$$

$$R_{19}=3\left[1-\frac{32(1-\nu)}{\lambda^2R^2}\right]\frac{R_{13}}{R_3}+R_{14}K_5-2R_6+\frac{64(1-\nu)}{\lambda^2R^2}+R_8,$$

$$R_{20}=5\left[1-\frac{48(1-\nu)}{\lambda^2R^2}\right]\frac{R_{10}}{R_{15}}+\frac{48(1-\nu)}{\lambda^2R^2}[1-4(1-\nu)]\frac{R_4}{R_3}+$$
$$+R_{17}K_7+R_7+R_{10}.$$

By limiting expansions (51.8) to terms containing ε^2, we calculate the magnitude of the contour stress σ_ϑ from the derived approximate values of the functions φ, ψ, H. The maximum value of this stress (at $\vartheta=0$) is represented as

$$(\sigma_\vartheta)_{\max}=pk,$$

where p is the tensile force, and k, the stress concentration factor, is determined by the equality

$$(1-\varepsilon)k=1+\frac{1}{R_0}\left\{2+\varepsilon R_2-\frac{2\varepsilon^2R_{11}}{R_0}-\right.$$
$$\left.-\frac{\varepsilon}{R_3}[8(1-\nu)R_4+3\varepsilon R_{13}]-\frac{10\varepsilon^2R_{16}}{R_{15}}\right\}.$$

Below the factor k is tabulated as a function of ν and the ratio b/a for $R/l=3$. The values of the same factor by the classical theory are included for comparison.

$\frac{b}{a}$	By classical theory	By couple-stress theory			
		ν			
		0	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$
$\frac{3}{4}\left(\varepsilon=\frac{1}{7}\right)$	3.(6)	2.7611	2.9193	2.9790	3.1114
$\frac{2}{3}\left(\varepsilon=\frac{1}{5}\right)$	4	2.9391	3.1233	3.1879	3.3407

As seen from this table, according to the couple-stress theory the stress concentration factor decreases. The decrease is the greater the smaller is ν. When $\nu = 1/3$, for example, it amounts to about 17 per cent for $b/a = 3/4$, and for $b/a = 2/3$ it amounts to some 20 per cent.

52. Remark Concerning the Application of Muskhelishvili's Method

In cases where the function

$$z = \omega(\zeta), \tag{52.1}$$

which realizes the conformal mapping of a given physical region onto a unit circle in the plane of the auxiliary variable ζ, is rational, the well-known Muskhelishvili method ([93], Chap. V) enables one to find the functions φ and ψ from the first condition of (45.35) or (45.37) in explicit form, viz. in terms of the given functions on L and the boundary values of the functions $\frac{\partial H}{\partial x}$ and $\frac{\partial H}{\partial y}$. This circumstance in turn makes it possible in these cases to eliminate the unknowns φ and ψ from relations (45.35) and (45.37), thereby reducing the fundamental problems of plane elasticity with couple-stresses to those of finding the single function H from a certain boundary condition. To illustrate this by an example, consider the second fundamental problem for the case of an elliptical hole.

In the transformed region conditions (45.37) are of the form

$$\left.\begin{aligned} &\varkappa\varphi(\sigma) - \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\overline{\varphi'(\sigma)} - \overline{\psi(\sigma)} - m\,\frac{\overline{\Phi'(\sigma)}}{\overline{\omega'(\sigma)}} + \frac{2i}{\overline{\omega'(\sigma)}} - \frac{\partial H}{\partial\bar{\sigma}} = \\ &\qquad\qquad\qquad\qquad\qquad\qquad = 2\mu(g_1 + ig_2), \\ &\operatorname{Im}\left\{(\varkappa+1)\frac{\varphi'(\sigma)}{\omega'(\sigma)} + \frac{i\lambda^2}{2}H\right\} = 2\mu g_0 \end{aligned}\right\} \tag{52.2}$$

with

$$\omega(\zeta) = R\left(\zeta + \frac{n}{\zeta}\right), \qquad R > 0, \qquad 0 \leqslant n < 1. \tag{52.3}$$

Assuming for simplicity that the elastic displacements u, v remain bounded at infinity, we have from the first condition of (52.2) (see N. I. Muskhelishvili [93], Sec. 83)

$$\varkappa\varphi(\zeta) = \frac{1}{\pi}\int\limits_{\gamma} \frac{1}{\overline{\omega'(\sigma)}}\,\frac{\partial H}{\partial\bar{\sigma}}\,\frac{d\sigma}{\sigma - \zeta} + G(\zeta) \qquad (|\zeta| > 1), \tag{52.4}$$

$$G(\zeta) = -\frac{\mu}{\pi i}\int\limits_{\gamma} \frac{g_1 + ig_2}{\sigma - \zeta}\,d\sigma. \tag{52.5}$$

From formula (52.4) we find

$$\varkappa\varphi'(\zeta)=\frac{1}{\pi}\int\limits_{\gamma}\frac{d}{d\sigma}\left[\frac{1}{\overline{\omega'(\sigma)}}\frac{\partial H}{\partial\bar{\sigma}}\right]\frac{d\sigma}{\sigma-\zeta}+G'(\zeta).$$

Substituting the preceding expression for $\varphi'(\zeta)$ in the second condition of (52.2), we find the required condition for H in the following form:

$$\frac{\lambda^2}{2}|\omega'(\sigma)|^2H+$$
$$+\frac{\varkappa+1}{\varkappa}\operatorname{Re}\left\{\overline{\omega'(\sigma)}\lim_{\zeta\to\sigma}\frac{1}{\pi i}\int\limits_{\gamma}\frac{d}{d\tau}\left[\frac{1}{\overline{\omega'(\tau)}}\frac{\partial H}{\partial\bar{\tau}}\right]\frac{d\tau}{\tau-\zeta}\right\}=\Omega(\sigma), \qquad (52.6)$$

where

$$\Omega(\sigma)=2\mu|\omega'(\sigma)|^2g_0+\frac{\varkappa+1}{\varkappa}\operatorname{Re}\{\overline{\omega'(\sigma)}\,iG'^{+}(\sigma)\}. \qquad (52.7)$$

The function $\psi(\zeta)$ is expressed, together with $\varphi(\zeta)$, explicitly in terms of the given function $2\mu(g_1+ig_2)$, the boundary values of $\frac{\partial H}{\partial t}$ and the first and second order derivatives of φ.

The second fundamental problem of couple-stress elasticity is thus reduced to that of determining the solution of the transformed Helmholtz equation (51.7) from the boundary condition (52.6). Condition (52.6) contains under the improper integral the boundary values of the second order derivatives of the unknown function $H(\xi, \eta)$.

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TO THE READER

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